Convergent Systems:

System Theoretic Aspects and Applications

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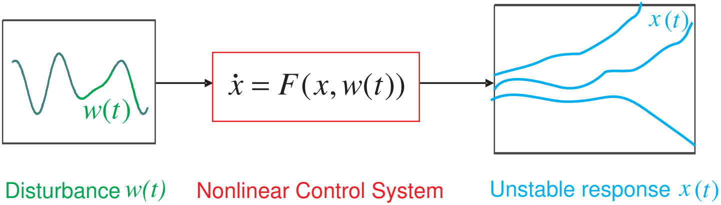
Outline

- What is convergence?
- Lyapunov characterisation + sufficient conditions
- Properties of convergent systems
- Related stability notions
- Applications (tracking, synchronization, output regulation, model reduction, steady-state analysis, extremum seeking, ...)
- Conclusions & Open issues

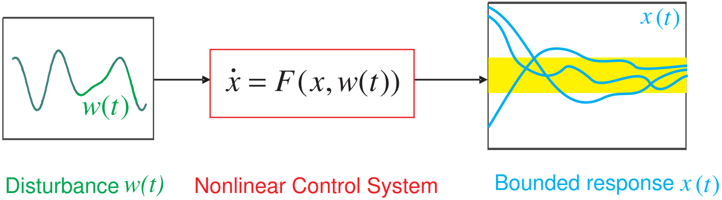
For the solutions of nonlinear systems with time-varying inputs we may have that:

- Solutions may grow unbounded for $t \to \infty$



For the solutions of nonlinear systems with time-varying inputs we may have that:

- Solutions may grow unbounded for $t \to \infty$
- Bounded input $\Rightarrow$ Bounded steady-states (e.g. Input-to-State Stability, Sontag, 1995)



Steady-state dynamics of systems with inputs

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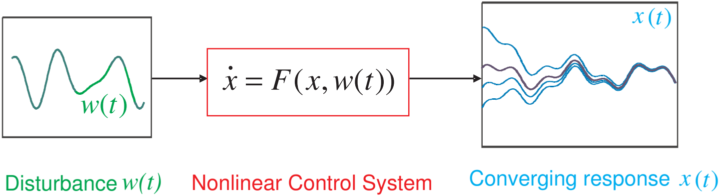
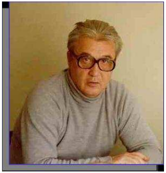
- Solutions may grow unbounded for $t \to \infty$
- Bounded input $\Rightarrow$ Bounded steady-states (e.g. Input-to-State Stability)
- Bounded input $\Rightarrow$ Unique bounded steady-state solution (Convergence)



A bit of History (Russia-1960's)

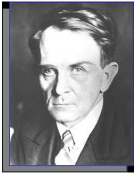
Unique steady-state response → Convergent Systems



V.A. Pliss
(Russia)

Fields of interest:
- Dynamics
- Stability theory

Known for his work on stability theory for dynamical systems



Boris Pavlovich Demidovich
(Russia, 1906-1977)

Fields of interest: Mathematics

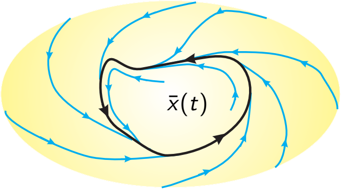
Known for his work on the stability of dynamical systems and in particular the convergence property

Vladimir A. Yakubovich
(Russia)

Fields of interest:
- Control theory
- Stability theory

Known for his work on absolute stability theory

- V.A. Pliss, B.P. Demidovich:Definition of convergent systems & sufficient conditions
- V.A.Yakubovich: Lur'e-type systems



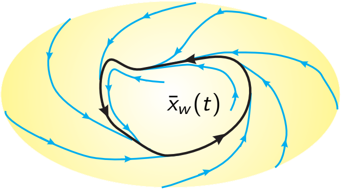
Definition
(according to Demidovich 1961,1967):
System $\dot{x} = F(x,t)$ is called

- **Convergent** if:
  1. there exists a solution $\bar{x}(t)$ defined and bounded for all $t \in \mathbb{R}$
  2. $\bar{x}(t)$ is globally asymptotically stable

- **Uniformly (exponentially) convergent** if additionally:
  - $\bar{x}(t)$ is globally uniformly asymptotically (exponentially) stable

- $\bar{x}(t)$ is called a steady-state solution
- $\bar{x}(t)$ is unique if the system is uniformly convergent

Convergent Systems: Definition



Definition
(according to Demidovich 1961,1967):
System $\dot{x} = F(x, w(t))$ is called

- **Convergent for a class of inputs** if:
  for any $w(t)$ from that class
  1. there exists a solution $\bar{x}_w(t)$ defined and bounded for all $t \in \mathbb{R}$
  2. $\bar{x}_w(t)$ is globally asymptotically stable

- **Uniformly (exponentially) convergent** if additionally:
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- $\bar{x}_w(t)$ is unique if the system is uniformly convergent

Convergent Systems: Examples

- Linear Systems:

$$\dot{x} = Ax + Bw(t)$$

- Global Asymptotic Stability (GAS) of $x = 0$ for $w(t) = 0$
  $\Longrightarrow$ Convergence for all bounded $w(t)$

  1. Unique solution $\bar{x}_w(t)$ bounded for $t \in [-\infty, \infty]$:

$$\bar{x}_w(t) = \int_{-\infty}^{t} e^{A(t-s)} Bw(s) ds$$

  2. $\bar{x}_w(t)$ is globally exponentially stable (due to Hurwitz A + linearity)

- For linear systems: GAS for the unperturbed system $\Rightarrow$
  1. Input-to-state stability: bounded steady-states
  2. Exponential convergence: a unique bounded steady-state

Convergent Systems: Examples

- Nonlinear Systems: Global Asymptotic Stability (GAS) of $x = 0$ for $w(t) = 0 \not\Longrightarrow$ Convergence for all bounded $w(t)$

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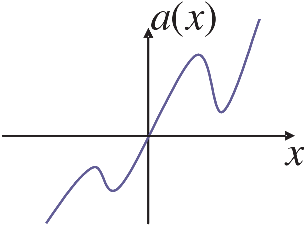
- Counter example:

$$\dot{x} = -a(x) + w, \quad x, w \text{ scalar}$$



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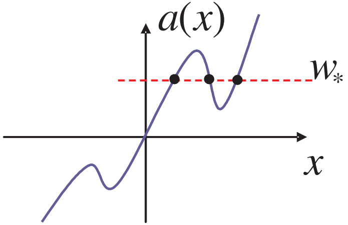

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2. For $w = w_*$: three equilibria $\Rightarrow$ not convergent!

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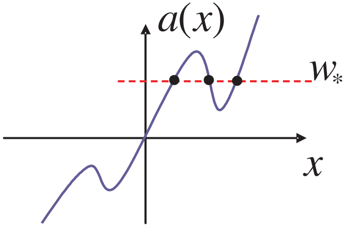

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2. For $w = w_*$: three equilibria $\Rightarrow$ not convergent!

Stronger conditions for convergence of nonlinear systems are needed

Lyapunov Characterisation (Rüffer, SCL2013)

Consider a system of the form $\dot{x} = f(t, x)$ and assume that

- f is continuous in (t, x) and $\mathcal{C}^1$ with respect to x
- the Jacobian $\frac{\partial}{\partial x} f(t, x)$ is bounded, uniformly in t

If system $\dot{x} = f(t, x)$ is globally uniformly convergent, then there exist a $V \in \mathcal{C}^1$, $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2, 3$, and $c \geq 0$ s.t.

$$\alpha_1(\|x - \bar{x}(t)\|) \leq V(t, x) \leq \alpha_2(\|x - \bar{x}(t)\|) \tag{1}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -\alpha_3(\|x - \bar{x}(t)\|) \tag{2}$$

$$V(t, 0) \leq c, \quad t \in \mathbb{R} \tag{3}$$

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Conversely, if $V \in \mathcal{C}^1$, $\alpha_i \in \mathcal{K}_\infty$, $i = 1,2,3$, and $c \geq 0$ are given such that for some trajectory $\bar{x}$ estimates (1)–(3) hold, then system $\dot{x} = f(t,x)$ must be globally uniformly convergent

Sufficient Conditions (Pavlov, Birkhäuser, 2005)

- Consider a perturbed nonlinear system

$$\dot{x} = f(x, w(t)), \quad f(0,0) = 0$$

- If there exist $V_1(x_1, x_2)$, $V_2(x) \in \mathcal{C}^1$, α_3, $\rho \in \mathcal{K}$ and α_1, α_2, α_4, $\alpha_5 \in \mathcal{K}_\infty$ such that
  1. we have incremental stability:

$$\alpha_1(\|x_1 - x_2\|) \leq V_1(x_1, x_2) \leq \alpha_2(\|x_1 - x_2\|)$$

$$\dot{V}_1 = \frac{\partial V_1}{\partial x_1} f(x_1, w) + \frac{\partial V_1}{\partial x_2} f(x_2, w) \leq -\alpha_3(\|x_1 - x_2\|)$$

  2. there exists a compact positively invariant set:

$$\alpha_4(\|x\|) \leq V_2(x) \leq \alpha_5(\|x\|)$$

$$\dot{V}_2 = \frac{\partial V_2}{\partial x} f(x, w) \leq 0, \text{ for } \|x\| \geq \rho(\|w\|)$$

  then the system is globally uniformly convergent

Sufficient Conditions (Special Cases)

- Consider the system:

$$\dot{x} = f(x, w(t)), \;\; x \in \mathbb{R}^n, \; w(t) \in \mathcal{PC}(\mathcal{W})$$

with $\mathcal{W} \subset \mathbb{R}^m$ and $\mathcal{PC}(\mathcal{W})$ bounded, piece-wise continuous inputs and with $|f(0, w)| \leq c < \infty$, $f(x, w)$ continuously differentiable with respect to x and continuous with respect to w

- Demidovich condition: (Demidovich, 1961,1967)

If there exist positive definite matrices $P = P^T > 0$ and $Q = Q^T > 0$ such that

$$P\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}^T P \leq -Q, \;\; \forall x \in \mathbb{R}^n, \; w \in \mathcal{W} \subset \mathbb{R}^m$$

then, the system is globally exponentially convergent for all bounded $w(t)$

Demidovich condition: Example

- Consider the system:

$$\dot{x}_1 = -x_1 + wx_2 + w$$
$$\dot{x}_2 = -wx_1 - x_2$$

- Jacobian of the vectorfield:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 & w \\ -w & -1 \end{bmatrix}$$

- Satisfies Demidovich condition with $P = I$ and $Q = 2I$:

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}^T = -2I < 0$$

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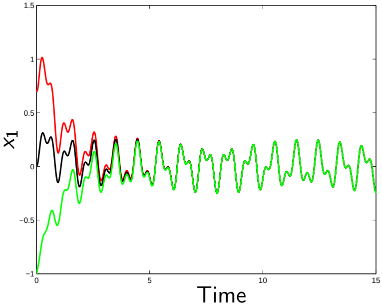
- Quasi-periodic excitation $w(t) = \sum_{i=1}^{2} A_i \sin(\omega_i t)$



Solutions converge to the same steady-state solution

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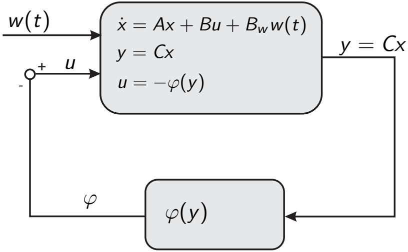
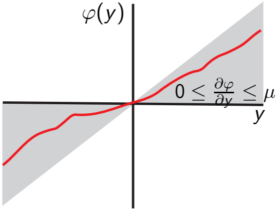
Sufficient Conditions (Special Cases)

- Consider a Lur'e-type system:

$$\dot{x} = Ax + Bu + B_w w(t)$$
$$u = -\varphi(Cx), \quad y, u \in \mathbb{R}$$



- **Circle-criterion-LIKE** condition (Yakubovich, 1964):
  If
  1. Incremental sector condition: $0 \leq \frac{\varphi(y_2)-\varphi(y_1)}{y_2-y_1} \leq \mu \; \forall y_1, y_2$
  2. $\mathrm{Re}\{C(j\omega I - A)^{-1}B\} \geq -\frac{1}{\mu}$
  3. A is Hurwitz



then the system is globally exponentially convergent with respect to $w(t)$

Convergence of Discrete-time Systems

Definition (Pavlov, van de Wouw, CDC2008, TAC2012):

- A nonlinear discrete-time system:

$$x[k+1] = f(x[k], k)$$

  is called uniformly (exponentially) convergent if
  - there exists a unique solution $\bar{x}[k]$ that is defined and bounded on $\mathbb{Z}$
  - $\bar{x}[k]$ is globally uniformly (exponentially) asymptotically stable
- The solution $\bar{x}[k]$ is called a steady-state solution

Convergence of Discrete-time Systems

Sufficient Condition (Pavlov, van de Wouw, CDC2008, TAC2012):

- Consider nonlinear discrete-time system:

$$x[k+1] = f(x[k], k)$$

  with a Lipschitz continuous right-hand side

- If
  - $|f(x_1, k) - f(x_2, k)|_P \leq \lambda |x_1 - x_2|_P$

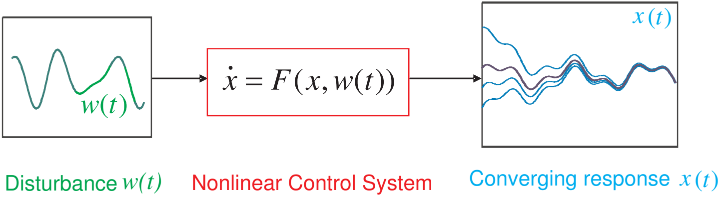
    for all $x_1, x_2 \in \mathbb{R}^n$, $k \in \mathbb{Z}$, $P = P^T > 0$ and $0 < \lambda < 1$
  - $\sup_{k \in \mathbb{Z}} |f(0, k)|_P =: C < +\infty$

  Then
  the system is globally exponentially convergent

Conditions for other classes of systems

- Other sufficient conditions for convergence:
  - Piecewise affine systems (Pavlov et. al., IJC2007, van de Wouw et. al. Automatica2008)
  - Measure differential inclusions (Leine, van de Wouw, IJBC2008)
  - Complementarity Systems (Camlibel, van de Wouw, CDC2007)
  - Switched Systems (van den Berg et. al., ADHS2006)
  - Interconnections of convergent systems and LMI-based conditions (Pavlov et. al., Birkhäuser 2005)
  - Time-varying Lyapunov conditions for convergence (Pogromski et. al., SCL2013)
  - Delay differential equations (Pola et. al. 2009, Chaillet et. al., CDC2013)

Properties of Convergent Systems



1. There exists a (unique) globally asymptotically stable solution, bounded for $t \in [-\infty, \infty]$

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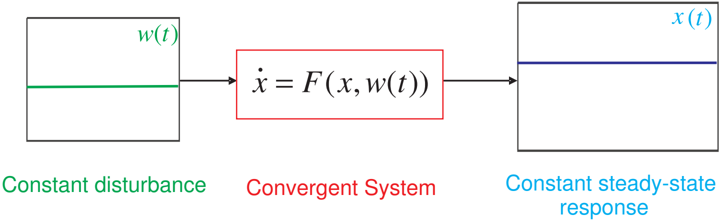
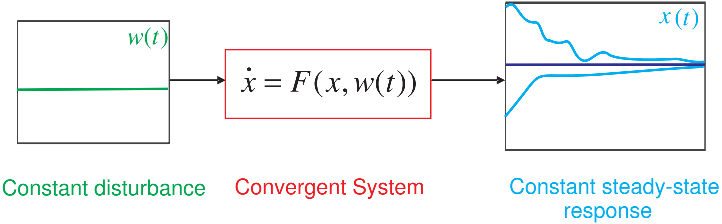


1. There exists a (unique) globally asymptotically stable solution, bounded for $t \in [-\infty, \infty]$
2. Constant inputs $\Rightarrow$ constant steady-state solutions



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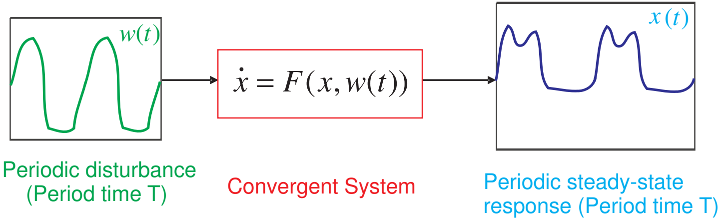

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3. Period-T inputs $\Rightarrow$ period-T steady-state solutions

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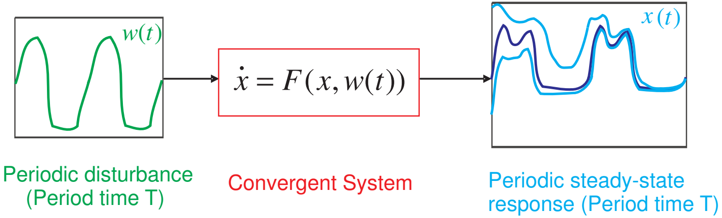

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2. Constant inputs $\Rightarrow$ constant steady-state solutions
3. Period-T inputs $\Rightarrow$ period-T steady-state solutions

Definition (Pavlov et. al., 2005):
System $\dot{x} = F(x, w(t))$ is called **Input-to-State Convergent** if:

1. it is uniformly convergent
2. for any $w(t)$ bounded for $t \in (-\infty, +\infty)$, it is input-to-state stable (ISS) with respect to the steady-state solution $\bar{x}_w(t)$,

   i.e. $\exists$ a $\mathcal{KL}$-function $\beta(r, s)$ and a $\mathcal{K}_\infty$-function $\gamma(r)$ such that any solution $x(t)$ corresponding to some perturbed input $\hat{w}(t) := w(t) + \Delta w(t)$ satisfies

   $$|x(t) - \bar{x}_w(t)| \leq \beta(|x(t_0) - \bar{x}_w(t_0)|, t - t_0) + \gamma(\sup_{t_0 \leq \tau \leq t} |\Delta w(\tau)|)$$

Input-to-State Convergence

- Input-state convergence is based on the notion of input-to-state stability
- Guarantees robustness of the steady-state behaviour in the face of perturbations to the time-varying input

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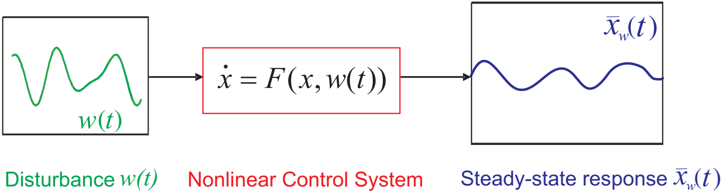

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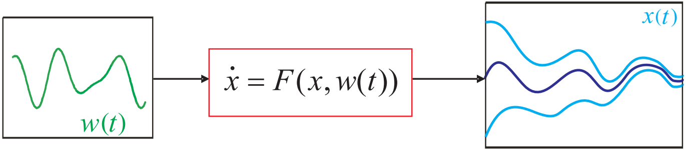

Disturbance $w(t)$ Nonlinear Control System Converging response $x(t)$

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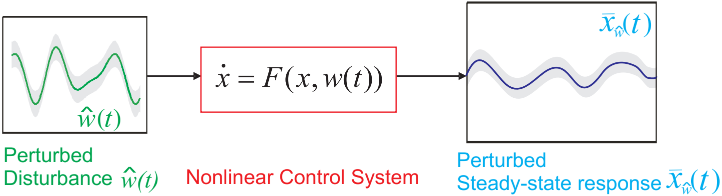

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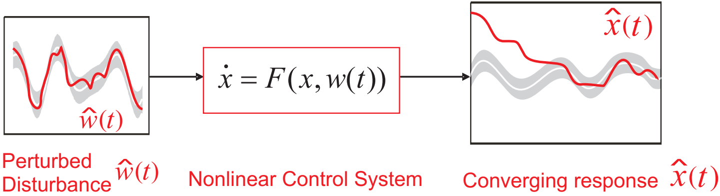
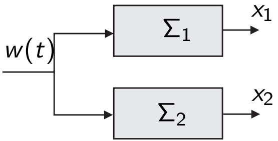


- Parallel connection of uniformly convergent systems is convergent (Pavlov et. al. Birkhäuser 2005)



Interconnection Properties of Convergent Systems

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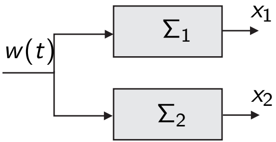
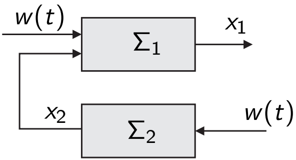


- Series connection of input-to-state convergent systems is input-to-state convergent (Pavlov et. al. Birkhäuser 2005)



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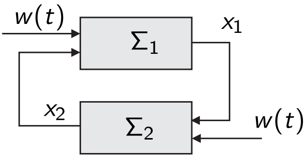
- Feedback connection of
  - an input-to-state convergent system Σ_1 and
  - a uniformly asymptotically stable system Σ_2

  is input-to-state convergent (Pavlov et. al. Birkhäuser 2005)



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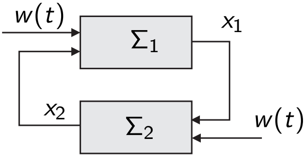

- Feedback connection of
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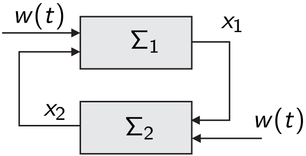
  is input-to-state convergent under additional small gain conditions (Besselink, TAC2012)



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Relation to Other Stability Notions

Early works:

- Global asymptotic stability of equilibria:
  Krasovskii (Russia, 1950s), Markus, Yamabe (U.S.A., 1960), Hartman (U.S.A., 1960,1962)

- Global asymptotic stability of time-varying (periodic) solutions:
  Borg (Sweden, 1960), Yoshizawa (Japan, 1966,1975), Smith (UK, 1986)

More recent works: Increase of interest since 1990-s (due to applications to observer design, synchronization, output regulation, tracking)

- Contraction:
  - Lohmiller, Slotine (Automatica1998), Jouffroy, Slotine (CDC2004), Zamani et. al. (TAC2011), Forni et. al. (TAC2013), Russo et. al. (PloS Comput. Biol. 2010), Sontag et. al., (CDC2014), ...
- Incremental stability:
  - Angeli (2002,2009), Fromion et. al. (1999), Zamani et.al. (TAC2011, SCL2013), Rüffer et. al. (SCL2013), Andrieu et. al. (CDC2013), Chaillet et. al. (CDC2013), ...

Convergence vs. Incremental Stability

Rüffer et. al. (CDC2012, SCL2013):

- Global Uniform Convergence $\not\Rightarrow$ Global Incremental Stability
- Global Uniform Convergence $\not\Leftarrow$ Global Incremental Stability
- Key differences:
  - Incremental stability does not imply the boundedness of solutions in forward time and the existence of a well-defined bounded steady-state solution
  - Convergence does not imply decay of the 'distance' between any two solutions uniform in the initial distance
- On compact sets: Convergence $\Leftrightarrow$ Incremental Stability
- Under additional conditions global uniform convergence implies global incremental stability and vice versa

Applications of convergence

- Steady-state analysis of nonlinear (control) systems using frequency response functions for nonlinear systems
- Controller design for tracking control, disturbance rejection or master-slave synchronisation
- Observer design
- Global output regulation
- Extremum seeking control for nonlinear systems with periodic steady states
- Stable inversion problem
- Model reduction for nonlinear systems with stability preservation and error bounds (BART)
- ...

Frequency Response Functions (FRFs)

- Linear Systems: $\dot{x} = Ax + Bw(t)$ $\quad G(j\omega) = (j\omega I - A)^{-1}B$

  State-space Model $\quad$ FRF

- FRF: a foundation for many powerful design and analysis tools

Frequency Response Functions (FRFs)

- Linear Systems: $\dot{x} = Ax + Bw(t)$ $G(j\omega) = (j\omega I - A)^{-1}B$

  State-space Model FRF

- FRF: a foundation for many powerful design and analysis tools

  QUESTION: Can we extend linear FRF to nonlinear systems?

- Linear Systems: $\dot{x} = Ax + Bw(t)$ $G(j\omega) = (j\omega I - A)^{-1}B$

  State-space Model FRF

- FRF: a foundation for many powerful design and analysis tools

  QUESTION: Can we extend linear FRF to nonlinear systems?

- Nonlinear systems: $\dot{x} = F(x, w(t))$ no transfer functions!

Frequency Response Functions (FRFs)

- Linear Systems:

Linear FRF $G(j\omega) = (j\omega I - A)^{-1}B$ characterizes all steady-state responses to harmonic excitations

$$w(t) = a\sin\omega t \Rightarrow \bar{x}^{a,\omega}(t) = \begin{bmatrix} ReG(j\omega) & ImG(j\omega) \end{bmatrix} \begin{bmatrix} a\sin\omega t \\ a\cos\omega t \end{bmatrix}$$

- Nonlinear systems: Possibly multiple steady-state solutions...

- Uniformly convergent systems: unique steady-state solution!

$$w(t+T) \equiv w(t) \Rightarrow \bar{x}_w(t+T) \equiv \bar{x}_w(t)$$

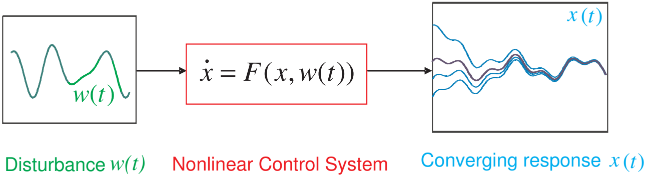
- Problem: Given a uniformly convergent system (with bounded inputs $w(t)$)

$$\dot{x} = F(x, w(t))$$

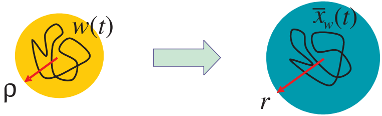
  can we find a function characterizing steady-state solutions to harmonic excitations?

- Answer: YES, but we need some additional assumptions...

Uniformly Bounded Steady State (UBSS)



- **Definition:** System has the *Uniformly Bounded Steady-State* if $\forall \rho > 0, \exists r > 0$ such that if $\sup_{t\in\mathbb{R}} \|w(t)\| \leq \rho$ then $\sup_{t\in\mathbb{R}} \|\bar{x}_w(t)\| \leq r$



- **Sufficient condition:** Input-to-state stability or Demidovich/Yakubovich conditions

Frequency Response Functions (FRFs)

- Theorem:

uniformly convergent system + uniformly bounded steady-state

↓

There is unique continuous $\chi(\omega, v_1, v_2)$ such that

$$\bar{x}_{a\omega}(t) = \chi(\omega, a\sin\omega t, a\cos\omega t)$$

- Definition: Function $\chi(\omega, v_1, v_2)$ is called the Frequency Response Function of the convergent system
- For details see (Pavlov, van de Wouw, Nijmeijer, TAC 2007)
- For discrete-time systems see (Pavlov, van de Wouw, TAC 2012)

- Consider the system:

$$\dot{x}_1 = -x_1 + x_2^2, \qquad \dot{x}_2 = -x_2 + w$$

Series connection of two uniformly convergent systems

- FRF can be calculated analytically:

$$\begin{aligned}\bar{x}_{a\omega}(t) =& \chi(\omega, v_1, v_2) = \chi(\omega, a\sin\omega t, a\cos\omega t)\\ =& \begin{bmatrix} c_1(\omega)v_1^2 + 2c_2(\omega)v_1v_2 + c_3(\omega)v_2^2 \\ b_1(\omega)v_1 + b_2(\omega)v_2 \end{bmatrix}\end{aligned}$$

with

$$b_1(\omega) = \frac{1}{1+\omega^2}, \quad b_2(\omega) = \frac{-\omega}{1+\omega^2}, \quad c_1(\omega) = \frac{2\omega^4+1}{\Delta(\omega)}$$

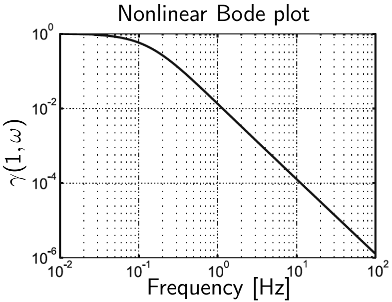
$$c_2(\omega) = \frac{\omega^3-\omega}{\Delta(\omega)}, \quad c_3(\omega) = \frac{2\omega^4+5\omega^2}{\Delta(\omega)}, \quad \Delta(\omega) = \frac{1+4\omega^2}{(1+\omega^2)^2}$$

FRFs: Example

- Steady-state output response ($y = h(x) = x_1$)

$$\bar{y}_{a\omega}(t) = h(\chi(\omega, v_1, v_2))$$

- Define the gain $\gamma(a, \omega) = \frac{1}{a}\left(\sup_{v_1^2+v_2^2=a^2} |h(\chi(\omega, v_1, v_2))|\right)$



Determination of nonlinear FRFs

- Tools have been developed to determine/compute these nonlinear FRFs for certain classes of systems, see

  Heertjes et.al. CST2006, van de Wouw, Automatica2008, Doris, ASME JDSMC2008, Pavlov et. al. CDC2007, CDC2008, Automatica2013

- Nonlinear FRFs have been exploited for the performance analysis and performance-based control design for industrial high-tech motion systems, such as wafer scanners, electron microscopes, pick and place machines, etc., see

  Heertjes et.al. CST2006, van de Wouw, Automatica2008, Pavlov et. al. Automatica2013, Hunnekens et. al. Mechatronics2014, Hunnekens et. al., IEEECST2015

Wafer scanners


Pick-and-place machines


Electron microscopes

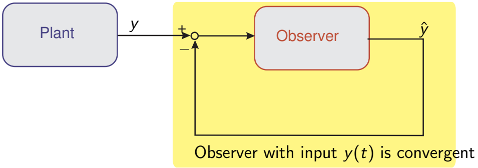

Printers, copiers

Convergence/incremental stability as a tool

... incremental stability questions are often reformulated as conventional stability questions for a suitable error system, the zero solution of the error system translating the convergence of two solutions to each other. This ad-hoc remedy may be successful in specific situations but it faces unpleasant obstacles that include both methodological issues - such as the issue of transforming a time-invariant problem into a time-variant one - and fundamental issues - such as the issue of defining a suitable error between trajectories.

Quote from Forni, F., Sepulchre, R. "Differential Lyapunov framework for contraction analysis" IEEE Transactions on Automatic Control, 59(3), 614-628, 2013.

Convergence-based observer design



- Plant + Observer:
- Nonlinear system:

$$\dot{x} = f(x), \qquad \text{measurement:} \;\; y = h(x)$$

- Observer:

$$\dot{\hat{x}} = f(\hat{x}) + l(y, \hat{y}), \;\; \hat{y} = h(\hat{x})$$

- Observer goal: the observer states should converge to the real plant state $\Rightarrow \hat{x} - x \to 0$ as $t \to \infty$
- Observer design: Choose $l(y, \hat{y})$ such that
  - Observer is a uniformly convergent system with input $y(t)$
  - $l(y, y) = 0$

Global Nonlinear Output Regulation

- Plant:

$$\begin{aligned} \dot{x} &= f(x, u, w) \\ e &= h_r(x, w), \qquad \text{Regulated output} \\ y &= h_m(x, w), \qquad \text{Measured output} \end{aligned}$$

- Exo-system: $\dot{w} = s(w), \quad w(0) \in \mathcal{W}$
  Assumption: For any $a > 0$ there exists $b > 0$ such that
  $|w(0)| \leq a \Rightarrow |w(t)| \leq b$ for all $t \in (-\infty, \infty)$

- Controller:

$$\begin{aligned} \dot{\xi} &= \eta(\xi, y) \\ u &= \theta(\xi, y) \end{aligned}$$

Global Nonlinear Output Regulation

Global Nonlinear Output Regulation Problem:

- (Very) loosely speaking: Find controller such that for all initial conditions

  1. $e \to 0$ for $t \to \infty$: Regulated output zeroing

  2. Bounded solutions of the closed-loop system with a well-defined globally asymptotically stable steady-state solution

Solvability of the Global Nonlinear Output Regulation Problem:

- Controller solves the global nonlinear output regulation problem

$$\Updownarrow$$ (Pavlov et. al. Birkhäuser 2005)

1. Controller renders the closed-loop system globally uniformly convergent with the UBSS property
2. There exist mappings π, σ satisfying the regulator equations:

$$\begin{aligned}\frac{\partial \pi}{\partial w} s(w) &= f(\pi(w), \theta(\sigma(w), h_m(\pi(w), w), w)\\ \frac{\partial \sigma}{\partial w} s(w) &= \eta(\sigma(w), h_m(\pi(w), w))\\ 0 &= h_r(\pi(w), w)\end{aligned}$$

Solvability of the Global Nonlinear Output Regulation Problem:

- Controller solves the global nonlinear output regulation problem

$$\Updownarrow$$ (Pavlov et. al. Birkhäuser 2005)

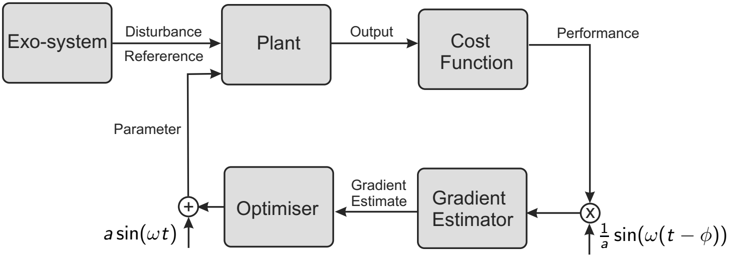
- Interpretation of conditions 1./2.:
  1. All solutions converge to a unique bounded steady-state solution
  2. There exists a feedforward generating a bounded steady-state solution on which the regulated output is zero

Disturbance Rejection on Experimental Motion Platform (Pavlov et. al. CST2007)

Extremum-seeking Control

- Extremum seeking control is an adaptive, data-based performance optimization strategy
- Most approaches are tailored to performance optimisation for steady-state equilibria (Krstic, Wang, TAC2000, Tan et. al. Automatica2006, ...)
- Problem: Extremum-seeking control for nonlinear systems with periodic steady state response
- For application to the performance optimization of variable-gain controllers for the control of motion stages in wafer scanners, see Hunnekens et. al. CDC2012, CST2015

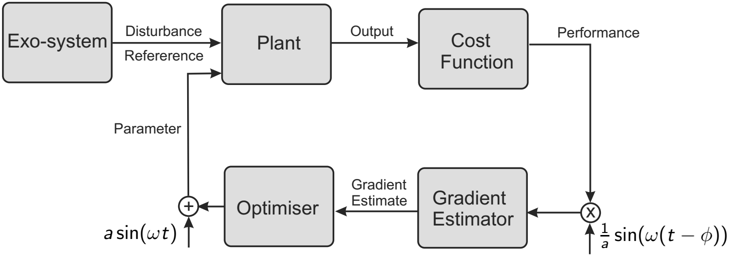
Extremum-seeking Control



Assumptions:

- Exo-system: produces bounded periodic disturbances with know period time
- Gradient Estimator + Optimiser with gain K: Standard
- Plant: Globally uniformly convergent
- Cost Function: E.g. based on $\mathcal{L}_p$ signal norm

Extremum-seeking Control



Result (van de Wouw et. al. CDC2012, Haring et. al. Automatica2013, Hunnekens et. al., CST2015):

- Optimal performance can be approached arbitrarily closely (for an arbitrarily large set of initial conditions) by tuning the controller parameters a, ω and $K/(a^2\omega)$ small enough
  More precisely: Closed-loop system is Semi-globally Practically Asymptotically Stable with parameters a, ω and $K/(a^2\omega)$

Example

- Plant:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -25x_1 - b(\theta)x_2 + w_1(t)$$
$$y = x_1$$
$$b(\theta) = 10 + 5(\theta - 10)^2$$
$$\theta : \text{Parameter}$$

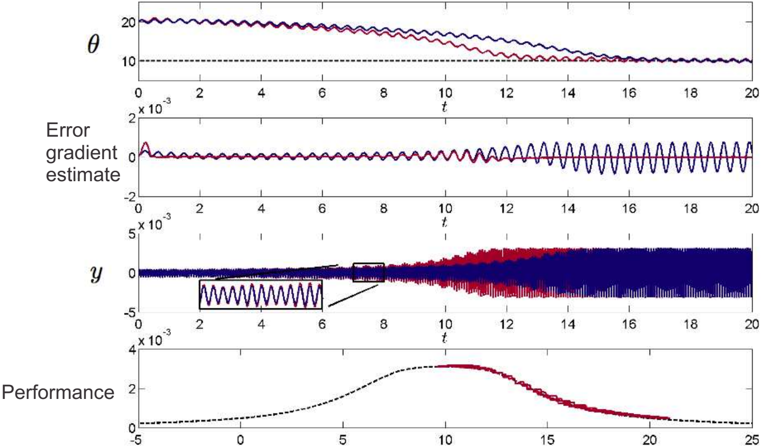
- Exosystem (Harmonic Oscillator):

$$\dot{w}_1 = 80w_2$$
$$\dot{w}_2 = -80w_1$$

- Cost Function: $L_\infty(y_d(t))$
  with $y_d(t)(\tau) = y(t+\tau)$ for all $\tau \in [-t_d, 0]$

$\Rightarrow$ Goal performance optimization: Maximize ‘amplitude’ of the steady-state output solution

Example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Moving average gradient estimator
Gradient estimator based on a low-pass filter

Conclusions

- Convergence is stability property on system level providing (for an entire class of inputs)
  - Well-defined bounded steady-state behavior
  - Global uniform asymptotic stability of this steady-state solution
- Powerful tool in many applications, such as performance analysis, output regulation, model reduction, extremum seeking, synchronization, etc.
- Engineering applications: Control of motion stages in wafer scanners and electron microscopes, Control of optical storage drives, Control of robots, Synchronisation of networks of neurons, etc.

Open Issues

- Convergence/Incremental stability of hybrid systems
- Convergence/Incremental stability of delay systems
- Further reduction conservatism of sufficient conditions

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