Model reduction of nonlinear systems using incremental system properties

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L2S, Supelec, Gif-sur-Yvette, France, 8 October 2015
Acknowledgements

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Practical engineering problems typically lead to complex, high-order dynamical models

Disadvantages of high-order models

- Difficult system analysis
- Time-consuming simulations
- Controller design infeasible

Solar panel  Robot  Chemical plant
Introduction

Practical engineering problems typically lead to complex, high-order dynamical models.

Nonlinearities often play an important role:
- Mechanical systems: friction, backlash, hysteresis
- Electrical systems: nonlinear components, electrostatics

Solar panel | Robot | Chemical plant
The model reduction problem

**Problem.** Given a dynamical system $\Sigma$, find a reduced-order system $\hat{\Sigma}$ that approximates its input-output behavior

$$
\Sigma : \begin{cases}
\dot{x} = f(x) + g(x)u \\
y = h(x)
\end{cases}
$$

with $x \in \mathbb{R}^n$, $n$ large

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\hat{\Sigma} : \begin{cases}
\dot{\xi} = \hat{f}(\xi) + \hat{g}(\xi)u \\
\hat{y} = \hat{h}(\xi)
\end{cases}
$$

with $\xi \in \mathbb{R}^k$, $k < n$ small
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**Objectives**

1. Preservation of system properties (in particular, stability)
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\[ \text{with } x \in \mathbb{R}^n, \text{ } n \text{ large} \]

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**Objectives**

1. Preservation of system properties (in particular, stability)
2. Bound on the reduction error $e = y - \hat{y}$ (e.g., $\|e\|_2 \leq \varepsilon \|u\|_2$)
Model reduction techniques

Objectives

1. Preservation of relevant stability properties
2. Error bound (a priori)

For linear systems, model reduction techniques satisfying 1. and 2. exist (e.g., balanced truncation [Moore, Glover] and extensions)
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Model reduction for nonlinear systems

- Balanced truncation for nonlinear systems [Scherpen, Fujimoto]
- Moment matching for nonlinear systems [Astolfi]
- Trajectory piecewise linear approximation [Rewieński, White]
- Proper orthogonal decomposition [Sirovich, Berkooz]
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Existing model reduction techniques for nonlinear systems do not generally satisfy 1. and 2.
Outline

Model reduction for nonlinear systems
1. A class of convergent nonlinear systems
2. Nonlinear systems with incremental gain or passivity properties
3. Incremental balanced truncation for nonlinear systems
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2. Nonlinear systems with incremental gain or passivity properties
3. Incremental balanced truncation for nonlinear systems

Incremental system properties are crucial in

1. the preservation of relevant stability properties;
2. the derivation of a priori error bounds
Problem setting and approach

Motivation

- Nonlinearities often act only locally
- Examples
  - Mechanical systems with friction, hysteresis
  - Systems with nonlinear actuator dynamics
  - Variable-gain controlled linear systems
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- Examples
  - Mechanical systems with friction, hysteresis
  - Systems with nonlinear actuator dynamics
  - Variable-gain controlled linear systems

Class of Lur’e-type systems is included (when $\Sigma_{nl}$ is static)
Problem setting and approach

Model reduction approach

- Reduction of high-order linear subsystem $\Sigma_{\text{lin}}$ only, taking into account inputs $(u, v)$ and outputs $(y, w)$
- Reconnect nonlinear subsystem $\Sigma_{\text{nl}}$
Model reduction approach

- Reduction of high-order linear subsystem $\Sigma_{\text{lin}}$ only, taking into account inputs $(u, v)$ and outputs $(y, w)$
- Reconnect nonlinear subsystem $\Sigma_{\text{nl}}$
- Allows for the use of existing model reduction techniques
- Computationally feasible
Definition. The operator $F : \mathcal{L}_m^\infty \to \mathcal{L}_n^\infty$ defined as

$$F u(t) := \bar{x}_u(t)$$

is said to be the steady-state operator of the uniformly convergent system $\dot{x} = f(x, u), \ x \in \mathbb{R}^n, \ u \in \mathbb{R}^m$
Definition. A system is input-to-state convergent (ISC) if it is globally uniformly convergent and

$$|x(t) - \bar{x}_u(t)| \leq \beta(|x(t_0) - \bar{x}_u(t_0)|, t - t_0) + \gamma \left( \sup_{t_0 \leq \tau \leq t} |\tilde{u}(\tau) - u(\tau)| \right)$$

holds for all $t \geq t_0$, with $\beta \in KL$ and $\gamma \in K_\infty$. 
**Definition.** A system is input-to-state convergent (ISC) if it is globally uniformly convergent and

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holds for all \( t \geq t_0 \), with \( \beta \in \mathcal{KL} \) and \( \gamma \in \mathcal{K}_\infty \).

**Lemma.** Let a system \( \dot{x} = f(x, u) \) be input-to-state convergent. Then, the steady-state operator is incrementally bounded as

\[ \|F u_2 - F u_1\|_\infty \leq \gamma(\|u_2 - u_1\|_\infty), \quad \|x\|_\infty = \sup_{\tau \in \mathbb{R}} |x(\tau)| \]
Small-gain theorem for ISC systems

\[ \Sigma_x: \dot{x} = f(x, z, u) \text{ ISC with } \gamma_{xz}, \gamma_{xu} \]

\[ \Sigma_z: \dot{z} = g(z, x, v) \text{ ISC with } \gamma_{zx}, \gamma_{zv} \]
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**Theorem.** The feedback interconnection is input-to-state convergent if there exist functions \( \rho_1, \rho_2 \) of class \( \mathcal{K}_\infty \) such that

\[
(id + \rho_1) \circ \gamma_{xz} \circ (id + \rho_2) \circ \gamma_{zx}(s) \leq s
\]

holds for all \( s \geq 0 \)
**Small-gain theorem for ISC systems**

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\]

holds for all $s \geq 0$

**Ingredients of proof**

1. Existence of a steady-state solution of the coupled system
2. Input-to-state stability with respect to the steady-state solution
Problem setting (cont’d.)

\[ \begin{align*} 
\Sigma_{\text{lin}} : \quad & \dot{z} = g(z, w) \\
\Sigma_{\text{nl}} : \quad & v = h(z) 
\end{align*} \]

**Assumptions**

**A1.** $\Sigma_{\text{lin}}$ is asymptotically stable (i.e., input-to-state convergent)
Problem setting (cont’d.)

\[ \Sigma_{\text{lin}} : \left\{ \begin{array}{l}
\dot{z} = g(z, w) \\
v = h(z)
\end{array} \right. \]

Assumptions

A1. \( \Sigma_{\text{lin}} \) is asymptotically stable (i.e., input-to-state convergent)
A2. \( \Sigma_{\text{nl}} \) is input-to-state convergent
A3. The output function is incrementally bounded as

\[ |h(z_2) - h(z_1)| \leq \chi_{vz}(|z_2 - z_1|) \]
Problem setting (cont’d.)

\[ \Sigma_{\text{lin}}: \begin{cases} \dot{z} = g(z, w) \\ v = h(z) \end{cases} \]

**Assumptions**

**A1.** $\Sigma_{\text{lin}}$ is asymptotically stable (i.e., input-to-state convergent)

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**A3.** The output function is incrementally bounded as

\[ |h(z_2) - h(z_1)| \leq \chi_{vz}(|z_2 - z_1|) \]

**Property.** The steady-state output operator $G_v w := h(\bar{z}_w)$ satisfies

\[ \|G_v w_2 - G_v w_1\|_{\infty} \leq \chi_{vz} \circ \gamma_{zw}(\|w_2 - w_1\|_{\infty}) \]
Problem setting (cont’d.)

\[ \Sigma_{\text{lin}} : \{ \dot{z} = g(z, w) \} \]
\[ v = h(z) \]

Assumptions

A4. \( \exists \rho_1, \rho_2 \in \mathcal{K}_\infty \) such that the small-gain condition

\[
(id + \rho_1) \circ \gamma_{xv} \circ \chi_{vz} \circ (id + \rho_2) \circ \gamma_{zw} \circ \chi_{wx}(s) \leq s, \quad \forall s \geq 0
\]

holds, i.e., \( \Sigma = \mathcal{I}(\Sigma_{\text{lin}}, \Sigma_{\text{nl}}) \) is input-to-state convergent
Assumptions

A5. $\hat{\Sigma}_{\text{lin}}$ is asymptotically stable

A6. The steady-state error operators $\mathcal{E}_i(u, v) = F_i(u, v) - \hat{F}_i(u, v)$ satisfy, for some $\varepsilon_{ij} \in \mathcal{K}_\infty$, $i \in \{y, w\}$, $j \in \{u, v\}$,

$$\|\mathcal{E}_i(u_2, v_2) - \mathcal{E}_i(u_1, v_1)\|_\infty \leq \varepsilon_{iu}(\|u_2 - u_1\|_\infty) + \varepsilon_{iv}(\|v_2 - v_1\|_\infty)$$
Problem setting (cont’d.)

\[
\begin{align*}
\Sigma_{\text{lin}} & \quad u \quad y \\
\Sigma_{\text{nl}} & \quad v \quad w
\end{align*}
\]

\[
\begin{align*}
\hat{\Sigma}_{\text{lin}} & \quad \hat{u} \quad \hat{y} \\
\hat{\Sigma}_{\text{nl}} & \quad \hat{v} \quad \hat{w}
\end{align*}
\]

Assumptions

A5. \( \hat{\Sigma}_{\text{lin}} \) is asymptotically stable

A6. The steady-state error operators \( \mathcal{E}_i(u, v) = F_i(u, v) - \hat{F}_i(u, v) \) satisfy, for some \( \varepsilon_{ij} \in \mathcal{K}_\infty, \ i \in \{y, w\}, \ j \in \{u, v\} \),

\[
\|\mathcal{E}_i(u_2, v_2) - \mathcal{E}_i(u_1, v_1)\|_\infty \leq \varepsilon_{iu}(\|u_2 - u_1\|_\infty) + \varepsilon_{iv}(\|v_2 - v_1\|_\infty)
\]

Linear model reduction techniques satisfying A5. and A6. exist, e.g., balanced truncation [Moore, Glover]
Theorem. Let $\Sigma = \mathcal{I}(\Sigma_{\text{lin}}, \Sigma_{\text{nl}})$ and $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_{\text{lin}}, \Sigma_{\text{nl}})$ satisfy Assumptions A1–A6. Then,

1. $\hat{\Sigma}$ is input-to-state convergent if $\exists \hat{\rho}_1, \hat{\rho}_2 \in \mathcal{K}_\infty$ such that

$$
(id + \hat{\rho}_1) \circ \chi_{vz} \circ \gamma_{zw} \circ (id + \hat{\rho}_2) \circ (\chi_{wx} \circ \gamma_{xv} + \varepsilon_{wv})(s) \leq s,
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for all $s \geq 0$
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1. $\hat{\Sigma}$ is input-to-state convergent if $\exists \hat{\rho}_1, \hat{\rho}_2 \in \mathcal{K}_\infty$ such that
   $$(\text{id} + \hat{\rho}_1) \circ \chi_{vz} \circ \gamma_{zw} \circ (\text{id} + \hat{\rho}_2) \circ (\chi_{wx} \circ \gamma_{xv} + \varepsilon_{wv})(s) \leq s,$$
   for all $s \geq 0$

2. If 1. is satisfied, then there exists $\varepsilon \in \mathcal{K}_\infty$ such that the steady-state output error is bounded as
   $$\|\delta \tilde{y}_u\|_\infty \leq \varepsilon(\|u\|_\infty),$$
   with $\delta \tilde{y}_u := \tilde{y}_u - \tilde{\hat{y}}_u$. Moreover, $\varepsilon(\cdot)$ can be expressed in terms of the incremental gains of $\Sigma_{\text{lin}}$ and $\Sigma_{\text{nl}}$ and the error bound on the linear subsystems $\varepsilon_{ij}$. 
Ingredients in the proof

- Input-to-state convergence provides bound on amplifications of steady-state errors going through the subsystems
- Small-gain theorem provides boundedness of steady-state errors in closed-loop
Proof and properties

Ingredients in the proof

- Input-to-state convergence provides bound on amplifications of steady-state errors going through the subsystems
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Properties

- Preservation of input-to-state stability
Proof and properties

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Properties

- Preservation of input-to-state stability
- Bound on the \textit{steady-state} error. Recall that the steady-state solution is 1. defined for any bounded input, and, 2. unique
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- A **priori** error bound, i.e., based only on properties of Σ
Proof and properties

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Properties

- Preservation of input-to-state stability
- Bound on the **steady-state** error. Recall that the steady-state solution is 1. defined for any bounded input, and, 2. unique
- **A priori** error bound, i.e., based only on properties of $\Sigma$
- Error bound holds for all nonlinear systems $\Sigma_{nl}$ satisfying the same input-to-state convergence gains
Incremental $\mathcal{L}_2$ gain and passivity

Incremental properties (by input-to-state convergence) of $\Sigma_{nl}$ crucial in obtaining the (a priori) error bound
**Incremental $\mathcal{L}_2$ gain and passivity**

Incremental properties (by input-to-state convergence) of $\Sigma_{nl}$ crucial in obtaining the (a priori) error bound

**Alternative system classes** (using dissipativity theory [Willems])

1. Incremental $\mathcal{L}_2$ gain [Romanchuk & James]. A system $\Sigma_{nl}$ has a bounded incremental $\mathcal{L}_2$ gain if $\exists$ a function $S$ such that

$$\dot{S}(x_2, x_1) \leq \gamma^2 |u_2 - u_1|^2 - |y_2 - y_1|^2$$
Incremental $\mathcal{L}_2$ gain and passivity

Incremental properties (by input-to-state convergence) of $\Sigma_{nl}$ crucial in obtaining the (a priori) error bound

**Alternative system classes** (using dissipativity theory [Willems])

2. Incremental passivity [Pavlov & Marconi]. A system $\Sigma_{nl}$ is incrementally passive if $\exists$ a storage function $S$ such that

$$\dot{S}(x_2, x_1) \leq (u_2 - u_1)^T(y_2 - y_1)$$
Incremental $\mathcal{L}_2$ gain and passivity

Incremental properties (by input-to-state convergence) of $\Sigma_{nl}$ crucial in obtaining the (a priori) error bound

Alternative system classes (using dissipativity theory [Willems])

- **Approach**: bounded real [Opdenacker & Jonckheere] or positive real [Desai & Pal, Harshavarhana et al.] balancing of $\Sigma_{lin}$

- **Properties**: preservation of bounded $\mathcal{L}_2$ gain or passivity through small-gain or passivity theorem and a priori error bound due to incremental properties [Besselink et al., 2013]
Example

Nonlinear beam example

- Linear beam model using Euler beam elements
- Nonlinear damping characteristic (damping force $v$)

\[
\Sigma_{nl} : \begin{cases} 
\dot{z} = -z - \sigma(z) + \kappa w \\
v = z 
\end{cases}
\]

with $\sigma(z)$ an arbitrary nondecreasing continuous function
Nonlinear beam example

- Linear beam model using Euler beam elements
- Nonlinear damping characteristic (damping force $v$)

\[ \Sigma_{nl} : \begin{cases} \dot{z} = -z - \sigma(z) + \kappa w \\ v = z \end{cases} \]

with $\sigma(z)$ an arbitrary nondecreasing continuous function

$\Sigma_{nl}$ has a bounded incremental $\mathcal{L}_2$ gain with gain $\kappa$, i.e. $\mu = \kappa$
Example: results (1)

Reduction of $\Sigma_{\text{lin}}$ ($n = 80$) to obtain $\hat{\Sigma}_{\text{lin}}$ of order $k = 4$ for $\kappa = 20$

\[
\begin{array}{c|c|c}
\kappa & \varepsilon \text{ (a priori)} & \varepsilon \text{ (a posteriori)} \\
20 & 0.903 \cdot 10^{-2} & 0.530 \cdot 10^{-5}
\end{array}
\]
Example: results (1)

Reduction of $\Sigma_{\text{lin}}$ ($n = 80$) to obtain $\hat{\Sigma}_{\text{lin}}$ of order $k = 4$ for $\kappa = 40$

<table>
<thead>
<tr>
<th>$\kappa$</th>
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<th>$\varepsilon$ (a posteriori)</th>
</tr>
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<tbody>
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<tr>
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</tr>
</tbody>
</table>
Example: results (1)

Reduction of $\Sigma_{\text{lin}} \ (n = 80)$ to obtain $\hat{\Sigma}_{\text{lin}}$ of order $k = 4$ for $\kappa = 60$

![Graph showing $|H_{\text{wv}}|$ and $|\hat{H}_{\text{wv}}|$ with $\kappa^{-1}$ and $\kappa^{-1} - \varepsilon_{\text{lin}}$](image)

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</tr>
<tr>
<td>60</td>
<td>--</td>
<td>$1.239 \cdot 10^{-5}$</td>
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</table>
Example: results (2)

Simulation of $\Sigma$ and $\hat{\Sigma}$ for $u(t) = 10^2 \text{sign}(\sin(2\pi 10t))$ and $\kappa = 60$

![Graph of output and nonlinearity](image)
Overview & Incremental balancing

Overview. Model reduction for ...

1. input-to-state convergent systems
2. systems with incremental dissipativity properties
Overview & Incremental balancing

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Properties

- Preservation of system properties and a priori error bound
- Computationally attractive
- Nonlinearity not explicitly taken into account

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Overview & Incremental balancing

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Properties

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◮ Computationally attractive
◮ Nonlinearity not explicitly taken into account

Objective. Incorporate nonlinearities in the reduction procedure

Incremental balanced truncation for models of the form

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)u, & x \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \ y \in \mathbb{R}^p \\ y = h(x) \end{cases}$$
Balanced truncation

\[ u \xrightarrow{} \Sigma \xrightarrow{} y \]

Observability and controllability function

\[ L_o(x_0) = \int_0^\infty |y(t)|^2 \, dt, \quad x(0) = x_0, \quad u = 0 \]

\[ L_c(x_0) = \inf_{u \in L^m_2} \int_{-\infty}^0 |u(t)|^2 \, dt, \quad u : x(-\infty) = 0 \leadsto x(0) = x_0 \]
Balanced truncation

\[ \Sigma \]

\[ \begin{align*}
L_o(x_0) &= \int_{0}^{\infty} |y(t)|^2 \, dt, \quad x(0) = x_0, u = 0 \\
L_c(x_0) &= \inf_{u \in \mathcal{L}_2^m} \int_{-\infty}^{0} |u(t)|^2 \, dt, \quad u : x(-\infty) = 0 \leadsto x(0) = x_0
\end{align*} \]

Linear systems (asymptotically stable)

\[ \begin{align*}
L_o(x) &= x^T Q x, \quad L_c(x) = x^T P^{-1} x
\end{align*} \]

1. **Balancing**: find transformation \( x = Tz \) such that
   \( L_o(z) = z^T \Sigma z, L_c(z) = z^T \Sigma^{-1} z \) with \( \Sigma = \text{diag}\{\sigma_1, \ldots, \sigma_n\} \)

2. **Truncation**: discard states corresponding to smallest \( \sigma_i \)'s
Balanced truncation

\[ u \xrightarrow{\Sigma} y \]

**Observability and controllability function**

\[ L_o(x_0) = \int_0^\infty |y(t)|^2 \, dt, \quad x(0) = x_0, u = 0 \]

\[ L_c(x_0) = \inf_{u \in L_2^m} \int_{-\infty}^0 |u(t)|^2 \, dt, \quad u : x(-\infty) = 0 \Rightarrow x(0) = x_0 \]

**Linear systems** (asymptotically stable)

- Preservation of asymptotic stability (when \( \sigma_k > \sigma_{k+1} \))
- A priori error bound of the form \( \|y - \hat{y}\|_2 \leq \varepsilon\|u\|_2 \) with

\[ \varepsilon = 2 \sum_{i=k+1}^n \sigma_i \]
Balanced truncation

\[
\begin{array}{c}
\Sigma \\
\end{array}
\]

\[u \rightarrow \Sigma \rightarrow y\]

**Observability and controllability function**

\[
L_o(x_0) = \int_0^\infty |y(t)|^2 \, dt, \quad x(0) = x_0, u = 0
\]

\[
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\]

**Nonlinear systems**

- Extension to nonlinear systems exist [Scherpen, Fujimoto]
- Preservation of local asymp. stability of \( x = 0 \) for \( u = 0 \)
- No error bound
Incremental observability function

\[
E_o(x_0, \bar{x}_0) = \sup_{u \in \mathcal{L}_2^m} \int_0^\infty |y(t) - \bar{y}(t)|^2 \, dt, \quad x(0) = x_0, \, \bar{x}(0) = \bar{x}_0
\]
Incremental observability function

\[ E_o(x_0, \bar{x}_0) = \sup_{u \in \mathcal{L}_2^m} \int_0^\infty |y(t) - \bar{y}(t)|^2 \, dt, \quad x(0) = x_0, \; \bar{x}(0) = \bar{x}_0 \]

Properties

- \( \Sigma \) is observable if and only if \( E_o(x_0, \bar{x}_0) > 0 \) for all \( x_0 \neq \bar{x}_0 \)
**Incremental observability function**

\[ \Sigma(x) \]

\[ \Sigma(x) \]

\[ u \]

\[ y - \hat{y} \]

\[ E_o(x_0, \bar{x}_0) = \sup_{u \in L^2_2} \int_0^\infty (y(t) - \tilde{y}(t))^2 \, dt, \quad x(0) = x_0, \bar{x}(0) = \bar{x}_0 \]

**Properties**

- \( \Sigma \) is observable if and only if \( E_o(x_0, \bar{x}_0) > 0 \) for all \( x_0 \neq \bar{x}_0 \)

- Related to incremental stability properties, i.e., if \( E_o(x_0, \bar{x}_0) > 0 \) for all \( x_0 \neq \bar{x}_0 \), then any two trajectories \( x(\cdot) \) and \( \bar{x}(\cdot) \) for a common input \( u(\cdot) \) satisfy

  \[ |x(t) - \bar{x}(t)| \leq \alpha(|x(0) - \bar{x}(0)|), \quad \forall t \geq 0, \alpha \in \mathcal{K} \]
Incremental observability function

\[ E_o(x_0, \bar{x}_0) = \sup_{u \in \mathcal{L}_2^m} \int_0^{\infty} |y(t) - \bar{y}(t)|^2 \, dt, \quad x(0) = x_0, \bar{x}(0) = \bar{x}_0 \]

**Properties**

- \( \Sigma \) is observable if and only if \( E_o(x_0, \bar{x}_0) > 0 \) for all \( x_0 \neq \bar{x}_0 \)
- Related to incremental stability properties

- **Linear systems**
  \[ E_o(x_0, \bar{x}_0) = (x_0 - \bar{x}_0)^T Q (x_0 - \bar{x}_0) = L_o(x_0 - \bar{x}_0) \]
Incremental controllability function

\[
E_c(x_0, \bar{x}_0) = \inf_{u, \bar{u} \in L^m_2} \int_{-\infty}^0 |u(t) + \bar{u}(t)|^2 \, dt, \quad u : 0 \sim x_0, \; \bar{u} : 0 \sim \bar{x}_0
\]
Incremental controllability function

\[ E_c(x_0, \bar{x}_0) = \inf_{u, \bar{u} \in L_2} \int_{-\infty}^{0} |u(t) + \bar{u}(t)|^2 \, dt, \quad u : 0 \rightsquigarrow x_0, \quad \bar{u} : 0 \rightsquigarrow \bar{x}_0 \]

Properties

- Related to reachability of \( \Sigma \)
- Related to boundedness of solutions (i.e., stability)
Incremental controllability function

\[ E_c(x_0, \bar{x}_0) = \inf_{u, \bar{u} \in L^2} \int_{-\infty}^{0} |u(t) + \bar{u}(t)|^2 \, dt, \quad u : 0 \rightarrow x_0, \quad \bar{u} : 0 \rightarrow \bar{x}_0 \]

Properties

- Related to reachability of \( \Sigma \)
- Related to boundedness of solutions (i.e., stability)
- Linear systems

\[ E_c(x_0, \bar{x}_0) = (x_0 + \bar{x}_0)^T P^{-1}(x_0 + \bar{x}_0) = L_c(x_0 + \bar{x}_0) \]
Incremental balancing

Assumption

1. $E_o$ and $E_c$ can be partitioned as

$$E_o(x, \bar{x}) = E^1_o(x_1, \bar{x}_1) + E^2_o(x_2, \bar{x}_2)$$
$$E_c(x, \bar{x}) = E^1_c(x_1, \bar{x}_1) + E^2_c(x_2, \bar{x}_2)$$

with $x^T = [x_1^T \ x_2^T]$, $\bar{x}^T = [\bar{x}_1^T \ \bar{x}_2^T]$

2. $E^2_o$ and $E^2_c$ satisfy, for some $\rho > 0$,

$$\frac{\partial E^2_o}{\partial \bar{x}_2}(x_2, 0) = -\rho^2 \frac{\partial E^2_c}{\partial \bar{x}_2}(x_2, 0)$$
Incremental balancing

Assumption

1. $E_o$ and $E_c$ can be partitioned as

$$E_o(x, \bar{x}) = E_o^1(x_1, \bar{x}_1) + E_o^2(x_2, \bar{x}_2)$$
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with $x^T = [x_1^T \ x_2]$, $\bar{x}^T = [\bar{x}_1^T \ \bar{x}_2]$

2. $E_o^2$ and $E_c^2$ satisfy, for some $\rho > 0$,

$$\frac{\partial E_o^2}{\partial \bar{x}_2}(x_2, 0) = -\rho^2 \frac{\partial E_c^2}{\partial \bar{x}_2}(x_2, 0)$$

Linear systems: the assumption is satisfied if the system is in balanced coordinates (then, $Q = P = \Sigma$ and $\rho = \sigma_n$)
Incremental balancing

Assumption

1. \( E_o \) and \( E_c \) can be partitioned as

\[
E_o(x, \bar{x}) = E_o^1(x_1, \bar{x}_1) + E_o^2(x_2, \bar{x}_2) \\
E_c(x, \bar{x}) = E_c^1(x_1, \bar{x}_1) + E_c^2(x_2, \bar{x}_2)
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with \( x^T = [x_1^T \ x_2] \), \( \bar{x}^T = [\bar{x}_1^T \ \bar{x}_2] \)

2. \( E_o^2 \) and \( E_c^2 \) satisfy, for some \( \rho > 0 \),

\[
\frac{\partial E_o^2}{\partial \bar{x}_2}(x_2, 0) = -\rho^2 \frac{\partial E_c^2}{\partial \bar{x}_2}(x_2, 0)
\]

Interpretation

- Existence of an "incrementally balanced" realization
- Can a coordinate transformation \( x = \phi(z) \) be found such that the assumption holds in the new coordinates \( z \)?
Partitioning in "incrementally balanced" form

\[ \Sigma : \begin{cases} 
\dot{x}_1 &= f_1(x_1, x_2) + g_1(x_1, x_2)u \\
\dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u \\
y &= h(x_1, x_2) 
\end{cases} \]
Incremental balanced truncation

Partitioning in "incrementally balanced" form

\[ \Sigma : \begin{cases} \dot{x}_1 &= f_1(x_1, x_2) + g_1(x_1, x_2)u \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u \\ y &= h(x_1, x_2) \end{cases} \]

Truncation, i.e., set \( x_2 = 0 \) and discard \( x_2 \)-dynamics

\[ \hat{\Sigma}_{n-1} : \begin{cases} \dot{\xi} &= f_1(\xi, 0) + g_1(\xi, 0)u \\ \hat{y} &= h(\xi, 0) \end{cases} \]

- One-step reduction
- \( \xi \in \mathbb{R}^{n-1} \) approximates \( x_1 \)
Incremental balanced truncation

Partitioning in "incrementally balanced" form

$$\Sigma : \begin{cases} 
\dot{x}_1 = f_1(x_1, x_2) + g_1(x_1, x_2)u \\
\dot{x}_2 = f_2(x_1, x_2) + g_2(x_1, x_2)u \\
y = h(x_1, x_2)
\end{cases}$$

Truncation, i.e., set $x_2 = 0$ and discard $x_2$-dynamics

$$\hat{\Sigma}_{n-1} : \begin{cases} 
\dot{\xi} = f_1(\xi, 0) + g_1(\xi, 0)u \\
\hat{y} = h(\xi, 0)
\end{cases}$$

- One-step reduction
- $\xi \in \mathbb{R}^{n-1}$ approximates $x_1$

Lemma. $\hat{E}_o$ and $\hat{E}_c$ of $\hat{\Sigma}_{n-1}$ satisfy the bounds

$$\hat{E}_o(\xi, \bar{\xi}) \leq E_o^1(\xi, \bar{\xi}), \quad \hat{E}_c(\xi, \bar{\xi}) \geq E_c^1(\xi, \bar{\xi})$$
Stability preservation

**Theorem.** Let $E_0(x, \bar{x}) > 0$ for all $x \neq \bar{x}$ and $E_c(x, x) > 0$ for all $x \neq 0$. Under the assumptions stated before

1. There exists $\hat{\mathcal{X}} \subseteq \mathbb{R}^{n-1}$ and $\mathcal{U} \subseteq \mathcal{L}_2^m([0, \infty))$ such that any $\xi(\cdot)$ corresponding to $\xi(0) \in \hat{\mathcal{X}}$ and $u(\cdot) \in \mathcal{U}$ is bounded

2. $\hat{\Sigma}_{n-1}$ is incrementally stable, i.e.,

   $$|\xi(t) - \bar{\xi}(t)| \leq \alpha(|\xi(0) - \bar{\xi}(0)|), \quad \forall t \geq 0$$

   with $\xi(\cdot)$ and $\bar{\xi}(\cdot)$ solutions to $\xi(0)$ and $\bar{\xi}(0)$ and input $u(\cdot)$

3. For any bounded input $u(\cdot) \in \mathcal{L}_2^m([0, \infty))$,

   $$\lim_{t \to \infty} |h(\xi(t), 0) - h(\bar{\xi}(t), 0)| = 0$$
Error bound

**Theorem.** Under the assumptions as before, the error bound

\[ \| y - \hat{y} \|_2 \leq 2\rho \| u \|_2 \]

holds for any \( u(\cdot) \in \mathcal{L}_2^m([0, \infty)) \) and zero initial conditions.
Error bound

**Theorem.** Under the assumptions as before, the error bound

$$\|y - \hat{y}\|_2 \leq 2\rho \|u\|_2$$

holds for any $u(\cdot) \in L^m_2([0, \infty))$ and zero initial conditions

**Proof** based on the storage function

$$V(x_1, x_2, \xi) = E^1_o(x_1, \xi) + E^2_o(x_2, 0) + \rho^2 (E^1_c(x_1, \xi) + E^2_c(x_2, 0)),$$

which satisfies

$$\dot{V}(x_1, x_2, \xi) \leq (2\rho)^2 |u|^2 - |y - \hat{y}|^2$$
Error bound

**Theorem.** Under the assumptions as before, the error bound

$$\|y - \hat{y}\|_2 \leq 2\rho \|u\|_2$$

holds for any $u(\cdot) \in L^m_2([0, \infty))$ and zero initial conditions.

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$$V(x_1, x_2, \xi) = E^1_o(x_1, \xi) + E^2_o(x_2, 0) + \rho^2(E^1_c(x_1, \xi) + E^2_c(x_2, 0)),$$

which satisfies

$$\dot{V}(x_1, x_2, \xi) \leq (2\rho)^2|u|^2 - |y - \hat{y}|^2$$

**Remarks**

- Reduction to arbitrary order by repeated application
- For linear systems, incremental balanced truncation is equivalent to balanced truncation
Generalized incremental balancing

Open problems

- Computation of $E_o$ and $E_c$ demanding
- Assumption on "incrementally balanced form" needed
Generalized incremental balancing

Open problems

- Computation of $E_o$ and $E_c$ demanding
- Assumption on "incrementally balanced form" needed

Generalized incremental energy functions

$$\tilde{E}_o(x, \bar{x}) = (x - \bar{x})^T \tilde{Q}(x - \bar{x}) \geq E_o(x, \bar{x})$$
$$\tilde{E}_c(x, \bar{x}) = (x + \bar{x})^T \tilde{R}(x + \bar{x}) \leq E_c(x, \bar{x})$$
Generalized incremental balancing

Open problems

- Computation of $E_o$ and $E_c$ demanding
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Generalized incremental energy functions

\[
\tilde{E}_o(x, \bar{x}) = (x - \bar{x})^T \tilde{Q}(x - \bar{x}) \geq E_o(x, \bar{x})
\]

\[
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\]

Properties

- Use generalized incremental energy functions (i.e., bounds)
- Results on stability and error bound hold
Generalized incremental balancing

Open problems

- Computation of $E_o$ and $E_c$ demanding
- Assumption on "incrementally balanced form" needed

Generalized incremental energy functions

$$\tilde{E}_o(x, \bar{x}) = (x - \bar{x})^T \tilde{Q}(x - \bar{x}) \geq E_o(x, \bar{x})$$
$$\tilde{E}_c(x, \bar{x}) = (x + \bar{x})^T \tilde{R}(x + \bar{x}) \leq E_c(x, \bar{x})$$

Properties

- Use generalized incremental energy functions (i.e., bounds)
- Results on stability and error bound hold

Generalized incremental balanced truncation provides a computationally feasible approach towards model reduction
Example: Nonlinear electronic circuit

Nonlinear electronic circuit (taken from [Rewieński])

- Nonlinear resistors $\eta$ with $\eta$ odd, nondecreasing and $\eta(0) = 0$
- Model $\Sigma$ with $f(x) = Ax + \varphi(x)$, $g(x) = B$, $h(x) = Cx$, and

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & \ddots \\ \vdots & \ddots & \ddots \\ 0 & \cdots & -2 & 1 \end{bmatrix}, \quad \varphi(x) = -\begin{bmatrix} \eta(x(1)) \\ \vdots \\ \eta(x(n)) \end{bmatrix}, \quad B = C^T = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$
Example: Results (1)

- The matrices $\tilde{Q}$ and $\tilde{R}$ can be chosen as
  
  $$\tilde{Q} = \tilde{R}^{-1} = \Sigma = \text{diag}\{\sigma_1, \ldots, \sigma_n\}, \quad \sigma_i > \sigma_{i+1} > 0,$$

  i.e., the system is in (generalized) incrementally balanced form

- Reduction from $n = 100$ to $k = 4$
Example: Results (2)

Simulations for $\eta(v) = \text{sign}(v)v^2$ and $u(t) = \frac{5}{2}(1 - \cos(2\pi\frac{1}{5}t))$ (left) and $u(t) = \frac{1}{2}(1 + \text{sign}(\sin(2\pi\frac{1}{20}t)))$ (right)

- A priori error bound of the form $\|y - \hat{y}\|_2 \leq \varepsilon\|u\|_2$ with
  \[ \varepsilon = 2 \sum_{i=5}^{100} \sigma_i = 3.401 \]
Conclusions & Open problems

Model reduction for nonlinear systems

1. A class of convergent nonlinear systems
2. Nonlinear systems with incremental gain or passivity properties
3. Incremental balanced truncation for nonlinear systems
Conclusions & Open problems

Model reduction for nonlinear systems

1. A class of convergent nonlinear systems
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Incremental system properties are crucial in

1. the preservation of relevant stability properties;
2. the derivation of a priori error bounds

References


Conclusions & Open problems

Model reduction for nonlinear systems

1. A class of convergent nonlinear systems
2. Nonlinear systems with incremental gain or passivity properties
3. Incremental balanced truncation for nonlinear systems

Incremental system properties are crucial in

1. the preservation of relevant stability properties;
2. the derivation of a priori error bounds

Open problems in model reduction

- Preservation of structure, e.g., in networks
- Computational methods for nonlinear systems