Stability of continuous-time quantum filters

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Background and motivation

Introduction to continuous-time filtering in classical and quantum cases

Stability of continuous-time quantum filters driven by a Wiener process

Design and stability of filters driven by both Poisson and Wiener processes with imperfections

Outline

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Some definitions

- **Notations :** Ket $|.\rangle$: a vector in the Hilbert space $\mathcal{H}$ and Bra $\langle .|$: a co-vector in the dual of the Hilbert space $\mathcal{H}$.
- **Observables :** Physical quantities like position, momentum, spin, etc : noncommutative counterparts of random variables.
  - Hermitian operators on the Hilbert space $\mathcal{H}$;
  - Spectral decomposition : $O = \sum_\mu \lambda_\mu P_\mu$, λ_μ are the eigenvalues and reals, and P_μ are the orthogonal projectors on the corresponding eigenspaces : $\sum P_\mu = I$.
- **States :** a summary of the status of a physical system that enables the calculation of statistical quantities associated with observables.
  - can be represented by a vector $|\psi\rangle$ (wave function) in the Hilbert space $\mathcal{H}$, with $\langle\psi|\psi\rangle = ||\psi||^2 = 1$.
  - or by a density matrix ρ : self-adjoint operator on $\mathcal{H}$; $\rho \geq 0$ and $\mathrm{Tr}(\rho) = 1$. (Noncommutative counterpart of a probability density.)

- **Pure state :** The unit vector $|\psi\rangle$ corresponds to a pure state. It is associated to the rank one projector $\rho = |\psi\rangle\langle\psi|$, which satisfies $\mathrm{Tr}(\rho) = \mathrm{Tr}(\rho^2) = 1$.

- **Mixed state :** The density operator takes the following form $\rho = \sum_\nu p_\nu |\psi_\nu\rangle\langle\psi_\nu|$, where p_ν is the probability that system is in the pure state $|\psi_\nu\rangle$. We have $\mathrm{Tr}(\rho) = 1$, however, $\mathrm{Tr}(\rho^2) < 1$.

- **Composite system :** Consider a system composed of two sub-systems A and B with states $|\psi^A\rangle$ and $|\psi^B\rangle$ which are represented in the Hilbert space $\mathcal{H}^A$ and $\mathcal{H}^B$. The joint state is $|\psi^A\rangle \otimes |\psi^B\rangle$ living in $\mathcal{H}^A \otimes \mathcal{H}^B$.

Measurements

- A measurement is a physical procedure that produces numerical results related to observables.
- The allowable results take values in the spectrum $\mathrm{spec}(A)$ of a chosen observable A.
- Given the state ρ, the value $\mu \in \mathrm{spec}(A)$ is observed with probability $\mathrm{Tr}\left(\rho P_\mu\right) \Longrightarrow$ the expectation of an observable A : $\langle A \rangle = \mathrm{Tr}\left(\rho A\right)$.
- **Projective measurements :** $\rho_+ = \frac{P_\mu \rho P_\mu}{p_\mu}$, with $p_\mu = \mathrm{Tr}\left(\rho P_\mu\right)$, $P_\mu^\dagger = P_\mu$ and $P_\mu^2 = P_\mu$
- **Positive operator values measure (POVM) :** $\rho_+ = \frac{M_\mu \rho M_\mu^\dagger}{\mathrm{Tr}\left(M_\mu \rho M_\mu^\dagger\right)}$, where $\sum_\mu M_\mu^\dagger M_\mu = I_S$ and the operators $M_\mu^\dagger M_\mu$ are positive.
- **Quantum non-demolition (QND) measurement :**$[H, O_s] = 0$.

Quantum commutative conditional expectation

Spectral theorem. Any commutative collection of operators can be simultaneously diagonalized. Denote the commutative algebra by $\mathcal{Y} = \mathrm{span}(P_y)$, where P_y are the orthogonal projections : $\sum P_y = I$. Denote $\mathcal{Y}'$ the collection of all operators that commute with $\mathcal{Y}$. Conditional expectation : an orthogonal projection from $\mathcal{Y}'$ to $\mathcal{Y}$:

$$
\begin{aligned}
\widehat{X} &= \mathbb{E}(X|\mathcal{Y}) \\
&= \sum_y \frac{P_y}{||P_y||} \langle \frac{P_y}{||P_y||}, X \rangle \\
&= \sum_y \frac{\mathbb{E}(P_y X)}{\mathbb{E}(P_y)} P_y .
\end{aligned}
$$

Here $\langle A, B \rangle = \mathrm{Tr}(\rho A^* B)$ and $||A|| = \sqrt{\langle A, A \rangle}$.

Quantum noncommutative conditional expectation

Spectral theorem is no more valid for non-commutative algebra $\mathscr{Y}$.

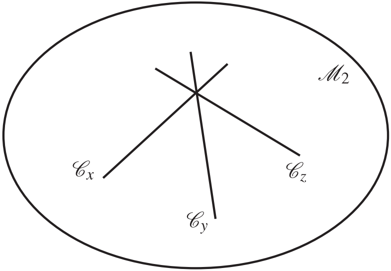


FIGURE: Example : non-commutative algebra $\mathscr{Y} = M_2$ (the space of complex 2×2 matrices) contains distinct quantum probability space determined by non-commutative observable σ_x, σ_y and σ_z.

Remark : The existence of a non-commutative conditional expectation is not always guaranteed ! (see e.g., M. Takesaki, Conditional expectations in von Neumann algebras. Journal of Functional Analysis 9(3), 306-321 (1972)).

Schrödinger equation

Consider a time-dependent system whose state at time t is described by the state $|\psi_t\rangle$ which evolves as follows

$$i\frac{d}{dt}|\psi_t\rangle = H(t)|\psi_t\rangle,$$

where H is a time-varying Hermitian operator ($H^\dagger = H$), called the Hamiltonian.
Take the initial state $|\psi_0\rangle$, then

$$|\psi_t\rangle = U_t|\psi_0\rangle,$$

with the linear operator U_t satisfies

$$i\frac{d}{dt}U_t = H(t)U_t, \quad U_0 = I. \Longrightarrow U_t^\dagger U_t = U_t U_t^\dagger = I.$$

The perfect discrete-time non-linear Markov model

Consider a finite-dimensional quantum system (the underlying Hilbert space $\mathcal{H} = \mathbb{C}^d$ is of dimension $d > 0$) being measured through a generalized measurement procedure at discrete-time intervals. [1]

$$\mathcal{D} := \{\rho \in \mathbb{C}^{d\times d} \mid \quad \rho = \rho^\dagger, \quad \mathrm{Tr}(\rho) = 1, \quad \rho \geq 0\}.$$

The random evolution of the state $\rho_k \in \mathcal{D}$ at time-step k is modeled through the following Markov process :

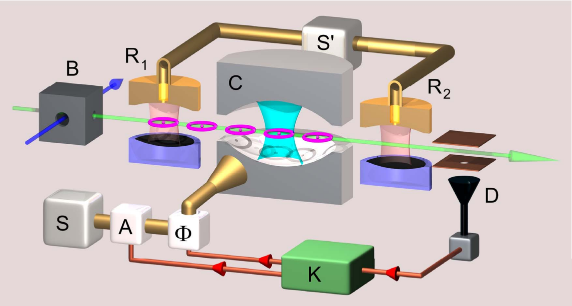
$$\rho_{k+1} = \mathbb{M}_{\mu_k}(\rho_k) := \frac{M_{\mu_k}\rho_k M_{\mu_k}{}^\dagger}{\mathrm{Tr}\left(M_{\mu_k}\rho_k M_{\mu_k}{}^\dagger\right)},$$

where,

- To each measurement outcome μ is attached the Kraus operator $M_\mu \in \mathbb{C}^{d\times d}$ depending on μ. We have $\sum_{\mu=1}^m M_\mu{}^\dagger M_\mu = I$.
- μ_k is a random variable taking values μ in $\{1, \cdots, m\}$ with probability $p_{\mu,\rho_k} = \mathrm{Tr}\left(M_\mu \rho_k M_\mu{}^\dagger\right)$.

1. S. Haroche and J.-M. Raimond. Exploring the quantum : atoms, cavities and photons. Oxford University Press, New York, 2006.

Example : LKB photon box

[2]

Experiment :
C. Sayrin et. al., Nature 477, 73-77, September 2011.

Theory :
I. Dotsenko et al., Physical Review A, 80 : 013805-013813, 2009.

R. Somaraju et al., Rev. Math. Phys., 25, 1350001, 2013.

H. Amini et. al., Automatica, 49 (9) : 2683-2692, 2013.

2. Courtesy of Igor Dotsenko

- $\mu_k \in \{g, e\}$;
- $d = n^{\max} + 1$;
- All operators are expressed in the truncated Fock-basis $(|n\rangle)_{n=0,\cdots,n^{\max}}$;
- $M_g = \cos(\frac{\phi_R + \phi_0(\mathbf{N}+\frac{1}{2})}{2})$, $M_e = \sin(\frac{\phi_R + \phi_0(\mathbf{N}+\frac{1}{2})}{2})$;
- $\mathbf{N} = \mathbf{a}^\dagger \mathbf{a}$ is the photon number operator ($\mathbf{N}|n\rangle = n|n\rangle$).

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Classical filtering

Consider the following stochastic dynamics

$$dX_t = v(X_t)\,dt + \sigma_X(X_t)dW_1,$$

with the following noisy observation

$$dY_t = h(X_t)\,dt + \sigma_Y(X_t)dW_2.$$

Filtering problem :

Obtain the dynamics for the least mean squares estimate for the state dynamics, i.e., $\pi_t(f) := \mathbb{E}\left(f(X_t)|\mathcal{F}_{t]}^Y\right)$, where $F_{t]}^Y$ is the $\sigma-$algebra generated by the observation processes up to time t.

Filter's dynamics [4]

Kallianpur-Striebel Formula [3] :

Take the Kallianpur-Striebel likelihood function

$$Z_t(X,Y) = \exp\left(\int_0^t h(X_s)^T dY_s - \frac{1}{2} h(X_s)^T h(X_s)\right).$$

The conditional expectation is given by

$$\pi_t(f) = \frac{\int f(x_t) Z_t P(dx)}{\int Z_t P(dx)} =: \frac{\sigma_t(f)}{\sigma_t(1)}.$$

3. Kallianpur, Striebel, 1968.
4. Davis and Marcus, 1981.

Filter's dynamics

Duncan-Motensen-Zakai equation :

Unnormalized filter satisfies

$$d\sigma_t(f) = \sigma_t(\mathcal{L}f)\,dt + \sigma_t(fh^T)dY_t.$$

Kushner-Stratonovich equation :

Normalized filter satisfies

$$d\pi_t(f) = \pi_t(\mathcal{L}f)\,dt + [\pi_t(fh^T) - \pi_t(f)\pi_t(h^T)]dI_t,$$

where I_t is the innovation process :

$$dI_t = dY_t - \pi_t(h)dt, \quad I(0) = 0.$$

Model of open quantum systems

Consider $G = (S, L, H)$, with unitary S describing **photon scattering phase**, L describing the **coupling** to the creation mode of the field and H describing the system **Hamiltonian.**

[Hudson and Parthasarathy, 1984.]

The evolution is described by evolution U with the following QSDE

$$dU(t) = \Big((S-I)d\Lambda(t) + dB^\dagger(t)L - L^\dagger S dB(t) - (\tfrac{1}{2}L^\dagger L + iH)dt\Big)\, U(t),$$
$$U(0) = I, \quad \text{where,}$$

- $dB(t) = B(t+dt) - B(t), \quad B(t)^\dagger = \int_0^t b^\dagger(s)ds$ and $B(t) = \int_0^t b(s)ds$;
- $[b(t), b^\dagger(s)] = \delta(t-s)$ and $\Lambda(t) = \int_0^t b^\dagger(s)b(s)ds$.

Model of open quantum systems

Quantum Ito rule : d(X Y)=dX Y+X dY+dX dY

Heisenberg-Langevin Equation :

The Heisenberg dynamics of an operator X is given by

$$j_t(X) = U^\dagger(t)(X \otimes I_{\text{field}})U(t)$$

$$\text{Using Itō rules :} \quad dj_t(X) = j_t(\mathcal{L}X)\,dt + dB^\dagger(t)j_t(S^\dagger[X,L]) + j_t([L^\dagger, X]S)dB(t) + j_t(S^\dagger XS - X)d\Lambda(t),$$

where the (Gorini-Kossakowski-Sudarshan-Lindblad) generator is $\mathcal{L}(X) = \frac{1}{2}[L^\dagger, X]L + \frac{1}{2}L^\dagger[X,L] - i[X,H]$

Input-Output relation :

The Output B_{out} is obtained from the input by

$$B_{\text{out}}(t) = U(t)^\dagger(I_{\text{system}} \otimes B(t))U(t)$$

$$\text{Using Itō rules :} \quad dB_{\text{out}}(t) = j_t(S)dB(t) + j_t(L)dt.$$

Derivation of quantum filters for Homodyne detection

Consider

$$dX(t) = -i[X(t), H(t)]dt + \mathcal{L}_L(X(t))dt + dB^\dagger(t)[X(t), L(t)] + [L^\dagger(t), X(t)]dB(t),$$

with $\mathcal{L}_L(X) = \frac{1}{2}L^\dagger[X, L] + \frac{1}{2}[L^\dagger, X]L$.

Take the quadrature measurement

$$Y(t) = B_{\text{out}}(t) + B^*_{\text{out}}(t).$$

- $\{Y(t) : t \geq 0\}$ is self-commuting : $[Y(t), Y(s)^T] = 0$ for all $t \geq s$, then, we can simultaneously diagonalize all observables ;
- We can estimate an observable that commutes with the observable $Y(t)$ up to time t : the non-demolition property, i.e.,

$$[X(\tau), Y(t)^T] = 0, \quad \text{for} \quad \tau \geq t.$$

Derivation of quantum filters in Homodyne detection

Recall :

$$Y(t) = B_{\mathrm{out}}(t) + B^*_{\mathrm{out}}(t),$$

then

$$dY(t) = (L(t) + L^*(t))dt + d(B(t) + B^*(t)).$$

Let $\mathcal{Y}(t)$ denotes the commutative subspace of operators generated by $Y(s), 0 \leq s \leq t$.

$$\hat{X}(t) = \pi_t(X) = \mathbb{E}\left(X(t)|\mathcal{Y}(t)\right).$$

Conditional characteristic approach :

$$c_f(t) = \exp\left(\int_0^t f(s)dY(s) - \frac{1}{2}\int_0^t |f(s)|^2 ds\right),$$

we have $dc_f(t) = f(t)c_f(t)dY(t)$, with $c_f(0) = I$.

Derivation of quantum filters for Homodyne detection

Suppose that $d\hat{X}(t) = \alpha(t)dt + \beta(t)dY(t)$, where α and β are to be determined from

$$\mathbb{E}\left(X(t)c_f(t)\right) = \mathbb{E}\left(\hat{X}(t)c_f(t)\right).$$

Filter's dynamics :

The best estimate satisfies :

$$\begin{aligned} d\pi_t(X) &= \pi_t(-i[X,H] + \mathcal{L}_L(X))dt \\ &+ \left(\pi_t(XL + L^\dagger X) - \pi_t(L + L^\dagger)\pi_t(X)\right)\left(dY(t) - \pi_t(L + L^\dagger)dt\right). \end{aligned}$$

Derivation of quantum filters for Homodyne detection (4)

A conditional density ρ may be defined by

$$\pi_t(X) = \mathrm{Tr}(\rho X).$$

Quantum filter in terms of the density operator :

The density operator-value stochastic process ρ satisfies the following

$$d\rho = \big(-i[H,\rho] + \mathcal{L}_L(\rho)\big)dt$$
$$+ (L\rho + \rho L^\dagger - \mathrm{Tr}\big((L+L^\dagger)\rho\big)\,\rho)(dY - \mathrm{Tr}\big((L+L^\dagger)\rho\big)\,dt)$$

Photon counting case

We measure the number observable $Y(t)$ given by

$$Y(t) = U(t)^{\dagger}\Lambda(t)U(t) = \Lambda_{\mathrm{out}}(t) = \int_0^t b_{\mathrm{out}}^{\dagger}(s)b_{\mathrm{out}}(s)ds.$$

Then,

Quantum filter for photon counting case :

$$d\rho(t) = -i[H,\rho]\,dt + \mathcal{L}_L(\rho)\,dt + \Gamma_L(\rho)dN(t),$$

where $\Gamma(\rho) = \frac{L\rho L^{\dagger}}{\mathrm{Tr}(\rho L^{\dagger}L)} - \rho$, and $dN(t) = dY - \mathrm{Tr}\left(\rho(t)L^{\dagger}L\right)\,dt$.

Quantum filtering

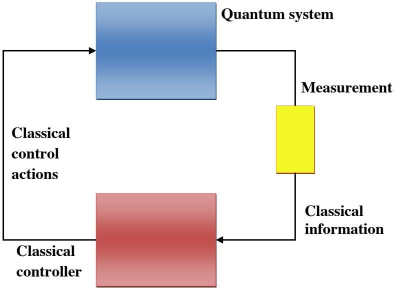
Take $\mathcal{Y}(t)$ as the $\sigma-$algebra generated by the observation process $(Y(s))_{0\leq s\leq t}$.Then,

- If $\mathcal{Y}$ is commutative, then the filtering problem is very similar to the classical one, e.g., : quadrature case using Homodyne detection, photon counting case ;
  [Davies, 1960s ; Belavkin, 1980s]
- If $\mathcal{Y}$ is non-commutative, we cannot apply the classical method to obtain the filter's equation.
  [No results]

Application of commutative filtering : measurement-based feedback

Estimation or filtering step consists of extracting **information** on the system state from the **past control input** and **measured output** values.

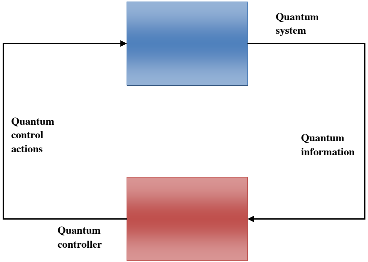
Computation of the next control input is based on the **current filter state** obtained in the first step.



Difficulty : any measurement necessarily modifies the system state.

[Belavkin, 1983 ; Wiseman and Milburn, 2010]

Application of non-commutative filtering : coherent quantum feedback



The controller is also a quantum system and information flowing in the feedback loop is also quantum (e.g. via a quantum field).
[Lloyd, 2000 ; James, Nurdin, Petersen, 2008 ; Mabuchi, 2008]

Outline

Some definitions

- **Asymptotic stability=Convergence :** The independence of the filter, after a long time, from the initial state estimate.
- **Observability** means that there do not exist two different initial states which give rise to measurement outcomes with the same probability.
- **Stability** means that a **distance** between the state and its associated quantum filter decreases.

The mathematical model

State space : density matrix

$$\mathcal{D} := \{\rho \in \mathbb{C}^{N\times N} | \quad \rho = \rho^\dagger, \quad \mathrm{Tr}(\rho) = 1, \quad \rho \geq 0\}.$$

Quantum filter dynamics :

$$\begin{aligned} d\rho &= \big(-i[H,\rho] + \mathcal{L}_L(\rho)\big)dt \\ &\quad + (L\rho + \rho L^\dagger - \mathrm{Tr}\left((L+L^\dagger)\rho\right)\rho)(dY - \mathrm{Tr}\left((L+L^\dagger)\rho\right)dt) \\ dy_t &= \mathrm{Tr}\left((L+L^\dagger)\rho_t\right)dt + dW_t. \end{aligned}$$

Quantum filter estimate equation : The estimate $\widehat{\rho} \in \mathcal{D}$ of quantum system follows [5]

$$\begin{aligned} d\widehat{\rho} &= \big(-i[H,\widehat{\rho}] + \mathcal{L}_L(\widehat{\rho})\big)dt \\ &\quad + (L\widehat{\rho} + \widehat{\rho} L^\dagger - \mathrm{Tr}\left((L+L^\dagger)\widehat{\rho}\right)\widehat{\rho})(dY - \mathrm{Tr}\left((L+L^\dagger)\widehat{\rho}\right)dt). \end{aligned}$$

What can we say about the distance between ρ and $\widehat{\rho}$?

5. A. Barchielli. Journal of the European Optical Society Part B, 1990.

The definition of the fidelity (distance)

We consider the following definition of fidelity between two density matrices ρ and σ :[6]

$$F(\rho,\sigma) = \left|\mathrm{Tr}\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)\right|^2.$$

$$\mathit{Distance} = 1 - F$$

- $F(\rho,\sigma) = 1$ iff $\rho = \sigma$;
- $F(\rho,\sigma) = 0$ means that the support of ρ and σ are orthogonal ;
- $F(\rho,\sigma)$ coincides with their inner product $\mathrm{Tr}\,(\rho\sigma)$ when at least one of the states ρ or σ is pure (i.e., orthogonal projector of rank one).

6. M. Nielsen and I. Chuang. Quantum Computation and quantum information, 1999.

Main Result

Theorem (Amini, Mirrahimi, and Rouchon, 2011)

Consider the Markov processes $(\rho_t, \widehat{\rho}_t)$ satisfying

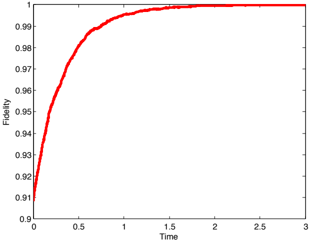
$$
\begin{aligned}
d\rho &= \big(-i[H,\rho] + \mathcal{L}_L(\rho)\big)\,dt \\
&\quad + \big(L\rho + \rho L^\dagger - \mathrm{Tr}\left((L+L^\dagger)\rho\right)\rho\big)\big(dY - \mathrm{Tr}\left((L+L^\dagger)\rho\right)dt\big) \\
dy_t &= \mathrm{Tr}\left((L+L^\dagger)\rho_t\right)dt + dW_t. \\
d\widehat{\rho} &= \big(-i[H,\widehat{\rho}] + \mathcal{L}_L(\widehat{\rho})\big)\,dt \\
&\quad + \big(L\widehat{\rho} + \widehat{\rho} L^\dagger - \mathrm{Tr}\left((L+L^\dagger)\widehat{\rho}\right)\widehat{\rho}\big)\big(dY - \mathrm{Tr}\left((L+L^\dagger)\widehat{\rho}\right)dt\big).
\end{aligned}
$$

respectively with initial states ρ_0, $\widehat{\rho}_0$ in $\mathcal{D}$. Then, the fidelity $F(\rho_t, \widehat{\rho}_t)$, is a submartingale, i.e. $\mathbb{E}\left(F(\rho_t,\widehat{\rho}_t) | (\rho_s,\widehat{\rho}_s)\right) \geq F(\rho_s,\widehat{\rho}_s)$, for all $t \geq s$.

Numerical test

Continuous homodyne measurement of a single qubit

$$H = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad L = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$



The average fidelity between the Markov processes ρ and $\widehat{\rho}$, over 500 realizations. Here the initial states are

$$\rho_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad \widehat{\rho_0} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}.$$

Sketch of proof

Proof : We proceed in two steps.

- In the first step, we describe how we obtain the stochastic master equations with Wiener processes as the limits of the stochastic master equations with Poisson processes. Then, we use a theorem by C. Pellegrini and F. Petruccione, 2009, to show the convergence of the solution of SMEs driven by Poisson processes towards the ones driven by Wiener processes.

- In the second step, we show that the fidelity between the real state and the quantum filter which are the solutions of stochastic master equations with Poisson processes is a submartingale. This can be done by discretizing SMEs driven by Poisson processes and using the stability results for discrete-time quantum non-linear Markov chain established by Rouchon, 2009.

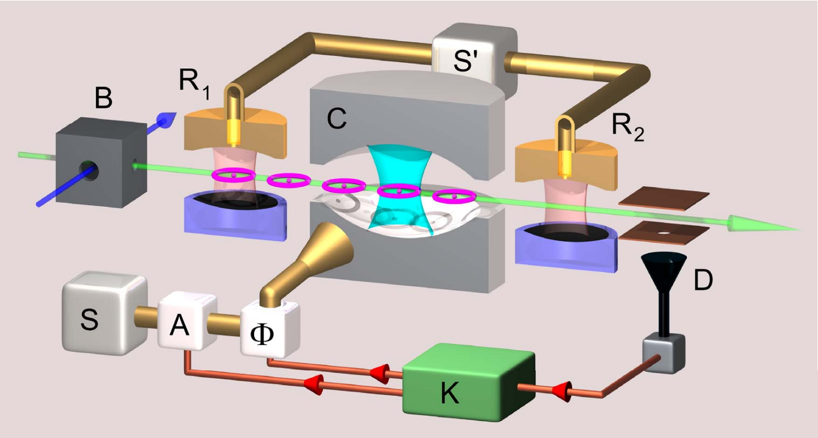
Further directions

- The considered fidelity has the potential to be used as a control Lyapunov function ;
- The fact that the fidelity between the real quantum state and the quantum-filter state increases in average remains valid for more general stochastic master equations where other Lindblad terms are added to $\mathcal{L}(\rho)$;
- The other extension of the problem is to consider the stochastic master equations driven by the Wiener processes and the jump processes at the same time and prove the stability of the quantum filter in this case (following section) ;
- Characterize the situations where the asymptotic convergence of such quantum filter is ensured.

Outline

Example : The closed-loop QED experiment [8]

[7]



Sampling time ($\sim$ 100 μs) long enough for feedback computations.

7. Courtesy of Igor Dotsenko
8. C. Sayrin et al., Nature, 1-September 2011

Imperfections in LKB experiment

Consider the ideal open-loop dynamics :

$$\rho_{k+1} = \begin{cases} \dfrac{M_g \rho_k M_g^\dagger}{\mathrm{Tr}\left(M_g \rho_k M_g^\dagger\right)} & y_k = g \text{ with probability } p_{g,k} = \mathrm{Tr}\left(M_g \rho_k M_g^\dagger\right) \\ \dfrac{M_e \rho_k M_e^\dagger}{\mathrm{Tr}\left(M_e \rho_k M_e^\dagger\right)} & y_k = e \text{ with probability } p_{e,k} = \mathrm{Tr}\left(M_e \rho_k M_e^\dagger\right) \end{cases}$$

Measurement Kraus operators $M_g = \cos\left(\frac{\phi_0 \mathbf{N} + \phi_R}{2}\right)$ and $M_e = \sin\left(\frac{\phi_0 \mathbf{N} + \phi_R}{2}\right)$: $M_g^\dagger M_g + M_e^\dagger M_e = \mathbb{1}$ with $\mathbf{N} = \mathbf{a}^\dagger \mathbf{a} = \mathrm{diag}(0, 1, 2, \ldots)$ the photon number operator.

Imperfections in LKB experiment include :

- Denote the detector efficiency by $\eta_d = 0.8$;
- The result of the measurement (atom in the state g or e) can be inter-changed : the fault rate $\eta_f = 0.1$;
- The measurement pulses can be empty of atom : the occupancy rate $\eta_a = 0.4$.

Markov chain with imperfections

Bayes' rule : $P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A)+P(B|A^c)P(A^c)}$.

- **Atom is detected in $|g\rangle$:**
  - Either the atom is in the state $|e\rangle$ and the detector detects it in the state $|g\rangle$ which arrives with the probability

$$P_g^f = \frac{\eta_f P_e}{\eta_f P_e + (1-\eta_f) P_g},$$

where $P_g = \mathrm{Tr}\left(M_g \rho M_g^\dagger\right)$ and $P_e = \mathrm{Tr}\left(M_e \rho M_e^\dagger\right)$.

  - Or the atom is really in the state $|g\rangle$, this happens with probability $1 - P_g^f$. Therefore,

$$\begin{aligned}\rho_{k+1} &= P_g^f \mathbb{M}_e(\rho_k) + (1-P_g^f)\mathbb{M}_g(\rho_k) \\ &= \frac{\eta_f M_e \rho_k M_e^\dagger + (1-\eta_f) M_g \rho_k M_g^\dagger}{\eta_f p_e + (1-\eta_f) P_g}.\end{aligned}$$

- **Atom is detected in $|e\rangle$:** The conditional evolution of the density matrix is given by

$$\rho_{k+1} = \frac{\eta_f M_g \rho_k M_g^\dagger + (1-\eta_f) M_e \rho_k M_e^\dagger}{\eta_f p_g + (1-\eta_f) P_e}.$$

- **No atom is detected :**
  - Either the pulse has been empty, which arrives with probability P_{na} given below,

$$P_{\mathrm{na}} = \frac{1-\eta_a}{\eta_a(1-\eta_d)+(1-\eta_a)} = \frac{1-\eta_a}{1-\eta_a\eta_d}.$$

  - or there has been an atom which has not been detected by the detector, this arrives with probability $1 - P_{\mathrm{na}}$. Then

$$\begin{aligned}\rho_{k+1} &= P_{\mathrm{na}}\rho_k + (1-P_{\mathrm{na}})(M_g \rho_k M_g^\dagger + M_e \rho_k M_e^\dagger) \\ &= \frac{(1-\eta_a)\rho_k + \eta_a(1-\eta_d)(M_g \rho_k M_g^\dagger + M_e \rho_k M_e^\dagger)}{1-\eta_a\eta_d}.\end{aligned}$$

General model : the new "observable" state $\widehat{\rho}$

- Take a set of Kraus operators M_q attached to the ideal detections for $q \in \{1, \cdots, m\}$: $\sum_{q=1}^{m} M_q^\dagger M_q = \mathbb{1}$, for some $m \in \mathbb{N}$.
- Assume that the real sensors provide an outcome $\mu' \in \{1, \cdots, m'\}$, for some $m' \in \mathbb{N}$.
- Suppose that we know the correlation between the events $\mu = q$ and $\mu' = p$ which is given through the stochastic matrix $\eta \in \mathbb{R}^{m' \times m}$:

$$\eta_{p,q} = \mathbb{P}(\mu' = p | \mu = q),$$

with $\eta_{p,q} \geq 0$ and for each q, $\sum_{p=1}^{m'} \eta_{p,q} = 1$.

General model : the new "observable" state $\widehat{\rho}$

- Assume $M_{q;k}$ denotes the Kraus operator corresponding to the $k-$th ideal measurement for $q \in \{1, ..., m\}$. We have

$$\mathbb{E}\left(\rho_{k+1} | \rho_k, \mu_k = q\right) = \mathbb{M}_{q;k}(\rho_k) := \frac{M_{q;k}\rho_k M_{q;k}^\dagger}{\operatorname{Tr}\left(M_{q;k}\rho_k M_{q;k}^\dagger\right)},$$

$$\text{with} \quad \mathbb{P}(\mu_k = q | \rho_k) = \operatorname{Tr}\left(M_{q;k}\rho_k M_{q;k}^\dagger\right).$$

- Let η^k be the stochastic matrix at step k. The optimal estimate is defined as $\widehat{\rho}_k = \mathbb{E}\left(\rho_k | (\rho_0, \mu_0', \cdots, \mu_{k-1}')\right)$.

General model : the new "observable" state $\widehat{\rho}$

Theorem (Somaraju et al., 2012.)

The optimal estimate $\widehat{\rho}_k$ satisfies the following recursive equation

$$\widehat{\rho}_{k+1} = \frac{\sum_{q=1}^{m} \eta_{p_k,q} M_{q;k} \widehat{\rho}_k M_{q;k}^{\dagger}}{\mathrm{Tr}\left(\sum_{q=1}^{m} \eta_{p_k,q} M_{q;k} \widehat{\rho}_k M_{q;k}^{\dagger}\right)},$$

if $\mu'_k = p_k$. Moreover

$$\mathbb{P}(\mu'_k = p_k | (\rho_0, \mu'_0, \cdots, \mu'_{k-1})) = \mathrm{Tr}\left(\sum_{q=1}^{m} \eta_{p_k,q} M_{q;k} \widehat{\rho}_k M_{q;k}^{\dagger}\right).$$

SDE driven by Poisson and/or Wiener processes

$$d\rho_t = \mathcal{L}(\rho_t)\,dt + \sum_{\nu=1}^{m_W} \Lambda_\nu(\rho_t)\,dW_t^\nu + \sum_{\mu=1}^{m_p} \gamma_\mu(\rho_t)\Big(dN_t^\mu - \mathrm{Tr}\left(C_\mu \rho_t C_\mu^\dagger\right)\,dt\Big),$$

where

- $\mathcal{L}(\rho_t) := -i[H,\rho_t] + \sum_{\mu=1}^{m_P} \mathcal{L}_\mu^P(\rho_t) + \sum_{\nu=1}^{m_W} \mathcal{L}_\nu^W(\rho_t)$, where
  $\mathcal{L}_\mu^P(\rho) := -\frac{1}{2}\{C_\mu^\dagger C_\mu, \rho\} + C_\mu \rho C_\mu^\dagger$,
  $\mathcal{L}_\nu^W(\rho) := -\frac{1}{2}\{L_\nu^\dagger L_\nu, \rho\} + L_\nu \rho L_\nu^\dagger$, and
  $\gamma_\mu(\rho) := \frac{C_\mu \rho C_\mu^\dagger}{\mathrm{Tr}\left(C_\mu \rho C_\mu^\dagger\right)} - \rho$, $\Lambda_\nu(\rho) := L_\nu \rho + \rho L_\nu^\dagger - \mathrm{Tr}\Big((L_\nu + L_\nu^\dagger)\rho\Big)\,\rho$.
- Detector click μ is related to the Poisson process $dN_t^\mu = N^\mu(t+dt) - N^\mu(t) = 1$ and happens with probability $\mathrm{Tr}\left(C_\mu \rho C_\mu^\dagger\right)\,dt$;
- Continuous detector ν refers to the Wiener process dW_t^ν by $dy_t^\nu = dW_t^\nu + \mathrm{Tr}\Big((L_\nu + L_\nu^\dagger)\rho_t\Big)\,dt$.

Heuristic approach : Jump processes

Perfect measurements

- Jump case : For any $\mu \neq 0$, define the following jump operators

$$M_\mu = \sqrt{dt}C_\mu.$$

The evolution is given by

$$\rho_{t+dt} = \frac{M_\mu \rho_t M_\mu^\dagger}{\mathrm{Tr}\left(M_\mu \rho_t M_\mu^\dagger\right)} = \frac{C_\mu \rho_t C_\mu^\dagger}{\mathrm{Tr}\left(C_\mu \rho_t C_\mu^\dagger\right)},$$

which happens with probability $\mathrm{Tr}\left(C_\mu \rho_t C_\mu^\dagger\right) dt$.

- Non jump case : For $\mu = 0$, we have

$$\sum_{\mu=1}^{m} M_\mu^\dagger M_\mu + M_0^\dagger M_0 = I.$$

we find $M_0 = I - \frac{1}{2}\sum_{\mu=1}^{m} C_\mu^\dagger C_\mu \, dt - iH\, dt$.

- We find

$$\rho_{t+dt} = \begin{cases} \frac{C_\mu \rho_t C_\mu^\dagger}{\mathrm{Tr}\left(C_\mu \rho_t C_\mu^\dagger\right)} \quad \text{with probability} \quad \mathrm{Tr}\left(C_\mu \rho_t C_\mu^\dagger\right) dt \\ \rho_t - \sum_{\mu=1}^m \frac{1}{2}\left(C_\mu^\dagger C_\mu \rho_t + \rho_t C_\mu^\dagger C_\mu\right) dt + \sum_{\mu=1}^m \mathrm{Tr}\left(C_\mu \rho_t C_\mu^\dagger\right) \rho_t \, dt - i[H, \rho_t] \, dt, \\ \text{with probability} \quad 1 - \sum_{\mu=1}^m \mathrm{Tr}\left(C_\mu \rho_t C_\mu^\dagger\right) dt. \end{cases}$$

- Now define m Poisson processes : $dN_t^\mu = N^\mu(t+dt) - N^\mu(t)$, which takes one with probability $\mathrm{Tr}\left(C_\mu \rho_t C_\mu^\dagger\right) dt$ and zero with probability $1 - \mathrm{Tr}\left(C_\mu \rho_t C_\mu^\dagger\right) dt$. Hence,

$$d\rho_t = \mathcal{L}(\rho_t)\, dt + \sum_{\mu=1}^{m} \Upsilon_\mu(\rho_t) \left(dN_t^\mu - \mathrm{Tr}\left(C_\mu \rho_t C_\mu^\dagger\right) dt\right).$$

Imperfect measurements.

- Set the ideal outcome $\mu \in \{0, \cdots, m\}$ and the real outcomes $\mu' \in \{0, \cdots, m'\}$;
- The Kraus operators attached to the ideal outcome μ are given by $M_\mu = \sqrt{dt} C_\mu$ and $M_0 = I - \frac{1}{2}\sum_{\mu=1}^{m} C_\mu^\dagger C_\mu \, dt - iH \, dt$;
- Take η as the stochastic matrix describing the correlation matrix between the events μ and μ';
- Suppose that η is well known with the following values :
    - for any $\mu' \neq 0$, $\eta_{\mu',0} = \bar{\eta}_{\mu'} \, dt$ and $\eta_{0,0} = 1 - \sum_{\mu'=1}^{m'} \bar{\eta}_{\mu'} \, dt$, with $\bar{\eta}_{\mu'} \geq 0$;
    - for any $\mu \neq 0$, $\eta_{0,\mu} = 1 - \sum_{\mu'=1}^{m'} \eta_{\mu',\mu}$, where $0 \leq \eta_{\mu',\mu} \leq 1$ and $\eta_{0,\mu} \geq 0$.
- We have for any $\mu' \in \{0, 1, \cdots, m'\}$:

$$\hat{\rho}_{t+dt} = \frac{\sum_{\mu=0}^{m} \eta_{\mu',\mu} M_\mu \hat{\rho}_t M_\mu^\dagger}{\mathrm{Tr}\left(\sum_{\mu=0}^{m} \eta_{\mu',\mu} M_\mu \hat{\rho}_t M_\mu^\dagger\right)},$$

$$\text{with probability} \quad \mathrm{Tr}\left(\sum_{\mu=0}^{m} \eta_{\mu',\mu} M_\mu \hat{\rho}_t M_\mu^\dagger\right).$$

- Finally, we find

$$\hat{\rho}_{t+dt} =$$

$$\begin{cases} \mathbb{L}_0(\hat{\rho}_t) & \text{with probability} \quad 1 - \sum_{\mu'=1}^{m'} \bar{\eta}_{\mu'} \, dt \sum_{\mu=1}^{m} \sum_{\mu'=1}^{m'} \eta_{\mu',\mu} \mathrm{Tr}\left(C_\mu^\dagger \hat{\rho}_t C_\mu \right) dt, \\ \mathbb{L}_{\mu'}(\hat{\rho}_t) & \text{with probability} \quad \bar{\eta}_{\mu'} \, dt + \sum_{\mu=1}^{m} \eta_{\mu',\mu} \mathrm{Tr}\left(C_\mu \hat{\rho}_t C_\mu^\dagger \right) dt, \end{cases}$$

where

$$\mathbb{L}_0(\hat{\rho}_t) = \hat{\rho}_t - \frac{1}{2} \sum_{\mu=1}^{m} \{ C_\mu^\dagger C_\mu, \hat{\rho}_t \} \, dt - i[H, \hat{\rho}_t] \, dt$$

$$+ \sum_{\mu=1}^{m} \left(1 - \sum_{\mu'=1}^{m'} \eta_{\mu',\mu} \right) C_\mu \hat{\rho}_t C_\mu^\dagger \, dt + \sum_{\mu=1}^{m} \sum_{\mu'=1}^{m'} \eta_{\mu',\mu} \mathrm{Tr}\left(C_\mu \hat{\rho}_t C_\mu^\dagger \right) \hat{\rho}_t \, dt$$

and

$$\mathbb{L}_{\mu'}(\hat{\rho}_t) = \frac{\bar{\eta}_{\mu'} \hat{\rho}_t + \sum_{\mu=1}^{m} \eta_{\mu',\mu} C_\mu \hat{\rho}_t C_\mu^\dagger}{\bar{\eta}_{\mu'} + \sum_{\mu=1}^{m} \eta_{\mu',\mu} \mathrm{Tr}\left(C_\mu \hat{\rho}_t C_\mu^\dagger \right)}.$$

- Now introduce m' Poisson processes $\widehat{N}_t^{\mu'}$ and the process $d\widehat{N}_t^{\mu'} := \widehat{N}_{t+dt}^{\mu'} - \widehat{N}_t^{\mu'}$ which takes one with probability $\bar{\eta}_{\mu'}\, dt + \sum_{\mu=1}^{m} \eta_{\mu',\mu} \mathrm{Tr}\left(C_\mu \widehat{\rho}_t C_\mu^\dagger\right) dt$ and zero with the complementary probability ;
- Hence, we have

$$d\widehat{\rho}_t = \mathcal{L}(\widehat{\rho}_t)\, dt + \sum_{\mu'=1}^{m'} \widehat{\Upsilon}_{\mu'}(\widehat{\rho}_t) \left(d\widehat{N}_t^{\mu'} - \bar{\eta}_{\mu'}\, dt - \sum_{\mu=1}^{m} \eta_{\mu',\mu} \mathrm{Tr}\left(C_\mu \widehat{\rho}_t C_\mu^\dagger\right) dt \right),$$

with $\mathcal{L}(\widehat{\rho}_t) = -i[H, \widehat{\rho}_t] - \frac{1}{2} \sum_{\mu=1}^{m} \{C_\mu^\dagger C_\mu, \widehat{\rho}_t\} + \sum_{\mu=1}^{m} C_\mu \widehat{\rho}_t C_\mu^\dagger$ and $\widehat{\Upsilon}_{\mu'}(\rho) = \frac{\bar{\eta}_{\mu'}\rho + \sum_{\mu=1}^{m} \eta_{\mu',\mu} C_\mu \rho C_\mu^\dagger}{\bar{\eta}_{\mu'} + \sum_{\mu=1}^{m} \eta_{\mu',\mu} \mathrm{Tr}\left(C_\mu \rho C_\mu^\dagger\right)} - \rho.$

SDE driven by both Poisson and/or Wiener processes including imperfections

Some notations :

- Imperfection model for the Poisson processes dN_t^{μ} :
  - real outcomes $\mu' \in \{0, 1, \cdots, m'_P\}$,
  - ideal outcomes $\mu \in \{1, \cdots, m_P\}$,
  - define $(m'_P + 1) \times m_P$ left stochastic matrix $\eta^P = (\eta^P_{\mu',\mu})_{0 \leq \mu' \leq m'_P, 1 \leq \mu \leq m_P}$,
  - define a positive vector $\bar{\eta}^P = (\bar{\eta}^P_{\mu'})_{1 \leq \mu' \leq m'_P}$ in $\mathbb{R}_+^{m'_P}$.
- Imperfection model for the diffusion processes dW_t^{ν} :
  - m'_W real continuous signals $y_t^{\nu'}$ with $\nu' \in \{1, \cdots, m'_W\}$,
  - m_W ideal continuous signals y_t^{ν} with $\nu \in \{1, \cdots, m_W\}$,
  - $m'_W \times m_W$ correlation matrix $\eta^W = (\eta^W_{\nu',\nu})_{1 \leq \nu' \leq m'_W, 1 \leq \nu \leq m_W}$, with $0 \leq \eta^W_{\nu',\nu} \leq 1$ and $\sum_{\nu'=1}^{m'_W} \eta^W_{\nu',\nu} \leq 1$.

SDE driven by both Poisson and/or Wiener processes including imperfections

Theorem (Amini, Pellegrini, and Rouchon, 2014)

$$d\widehat{\rho} = \mathcal{L}(\widehat{\rho}_t)\,dt + \sum_{\nu'=1}^{m'_W} \sqrt{\bar{\eta}^W_{\nu'}}\,\widehat{\Lambda}_{\nu'}(\widehat{\rho}_t)\,d\widehat{W}^{\nu'}_t$$

$$+ \sum_{\mu'=1}^{m'_P} \widehat{\gamma}_{\mu'}(\widehat{\rho}_t)\left(d\widehat{N}^{\mu'}_t - \bar{\eta}^P_{\mu'}\,dt - \sum_{\mu=1}^{m_P} \eta^P_{\mu',\mu}\mathrm{Tr}\left(C_\mu \widehat{\rho}_t C^\dagger_\mu\right)\,dt\right),$$

- $\bar{\eta}^W_{\nu'} = \sum_{\nu=1}^{m_W} \eta^W_{\nu',\nu}$, $\widehat{\gamma}_{\mu'}(\rho) := \frac{\bar{\eta}^P_{\mu'}\rho + \sum_{\mu=1}^{m_P} \eta^P_{\mu',\mu} C_\mu \rho C^\dagger_\mu}{\bar{\eta}^P_{\mu'} + \sum_{\mu=1}^{m_P} \eta^P_{\mu',\mu}\mathrm{Tr}(C_\mu \rho C^\dagger_\mu)} - \rho$,
  $\widehat{\Lambda}_{\nu'}(\rho) = \widehat{L}_{\nu'}\rho + \rho\widehat{L}^\dagger_{\nu'} - \mathrm{Tr}\left((\widehat{L}_{\nu'} + \widehat{L}^\dagger_{\nu'})\rho\right)\rho$, $\widehat{L}_{\nu'} := (\sum_{\nu=1}^{m_W} \eta^W_{\nu',\nu} L_\nu)/\bar{\eta}^W_{\nu'}$,
- the jump detector corresponds to $\widehat{N}^{\mu'}(t)$: $d\widehat{N}^{\mu'}_t = \widehat{N}^{\mu'}(t+dt) - \widehat{N}^{\mu'}(t) = 1$ happens with probability $\bar{\eta}^P_{\mu'} + \sum_{\mu=1}^{m_P} \eta^P_{\mu',\mu}\mathrm{Tr}\left(C_\mu \rho C^\dagger_\mu\right)$,
- the continuous detector ν' refers to $\widehat{y}^{\nu'}_t$ and $d\widehat{W}^{\nu'}_t$:
  $d\widehat{y}^{\nu'}_t = d\widehat{W}^{\nu'}_t + \sqrt{\bar{\eta}^W_{\nu'}}\mathrm{Tr}\left((\widehat{L}_{\nu'} + \widehat{L}^\dagger_{\nu'})\widehat{\rho}_t\right)\,dt$.

Stability results

The estimate filter $\widehat{\rho}^e$ has the following form

$$d\widehat{\rho}^e = \mathcal{L}(\widehat{\rho}^e_t)\,dt + \sum_{\nu'=1}^{m'_W} \sqrt{\bar{\eta}^W_{\nu'}}\widehat{\Lambda}_{\nu'}(\widehat{\rho}^e_t)\left(d\widehat{y}^{\nu'}_t - \sqrt{\bar{\eta}^W_{\nu'}}\mathrm{Tr}\left((\widehat{L}_{\nu'} + \widehat{L}^\dagger_{\nu'})\widehat{\rho}^e_t\right)dt\right)$$
$$+ \sum_{\mu'=1}^{m'_P} \widehat{\gamma}_{\mu'}(\widehat{\rho}^e_t)\left(d\widehat{N}^{\mu'}_t - \bar{\eta}^P_{\mu'}\,dt - \sum_{\mu=1}^{m_P} \eta^P_{\mu',\mu}\mathrm{Tr}\left(C_\mu \widehat{\rho}^e_t C^\dagger_\mu\right)dt\right),$$

Theorem (Amini, Pellegrini and Rouchon, 2014)

The fidelity $F(\widehat{\rho}_t, \widehat{\rho}^e_t)$ is a $(\mathcal{F}_t)-$submartingale, where $\mathcal{F}_t = \sigma\{(\widehat{\rho}_\tau, \widehat{\rho}^e_\tau)|\tau \leq t\}$. In particular, we have,

$$\mathbb{E}\left(F(\widehat{\rho}_\tau, \widehat{\rho}^e_\tau)|\mathcal{F}_t\right) = \mathbb{E}\left(F(\widehat{\rho}_\tau, \widehat{\rho}^e_\tau)|(\widehat{\rho}_t, \widehat{\rho}^e_t)\right) \geq F(\widehat{\rho}_t, \widehat{\rho}^e_t),$$

for all $\tau \geq t$.

Concluding remarks

- In ”Amini, Pellegrini, Rouchon, Russian Journal of Mathematical Physics, 2014”, we have shown rigorously, by applying quantum repeated measurement approach introduced by Attal and Pautrat, and by Gough, that imperfect discrete-time Markov chain converges to the continuous-time dynamics driven by both Poisson and Wiener processes.
- We have shown the stability of the filters taking into account measurement imperfections using the stability result for discrete-time filters which take into account imperfections.
- Difficult issue : Characterize the situations when we can conclude the convergence of the filters.
- Open problem : Filtering when the observation processes are non-commutative is an open problem. In ”Amini, Miao, Pan, James, and Mabuchi, On the generalization of linear least mean squares estimations to quantum systems with non-commutative outputs, EPJ Quantum Technology, 2015”, we extend Kalman filtering to non-commutative outputs.

Thank you !