Poisson INAR processes with serial and seasonal correlation

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Outline

- Integer valued autoregression and INAR(1) model
- Comparison of AR, INAR, and branching processes
- The purely seasonal INAR(1) model
- Estimation methods
- Simulation and real data examples
- INAR process with serial and seasonal correlation
- Stationarity and second order properties
- Estimation methods

Integer valued autoregression (INAR)

INAR(1) model (Al-Osh and Alzaid (1987))

$$X_t = \sum_{j=1}^{X_{t-1}} \xi_{t,j} + \varepsilon_t, \qquad t \in \mathbb{Z},$$

$\{\xi_{t,j}, : t \in \mathbb{Z}, j \in \mathbb{N}\}$ and $\{\varepsilon_t : t \in \mathbb{Z}\}$ are independent, non-negative, integer-valued, identically distributed r.v.'s

$\mathrm{P}(\xi_{1,1} \in \{0,1\}) = 1$, i.e., $\xi_{1,1}$ has Bernoulli distribution

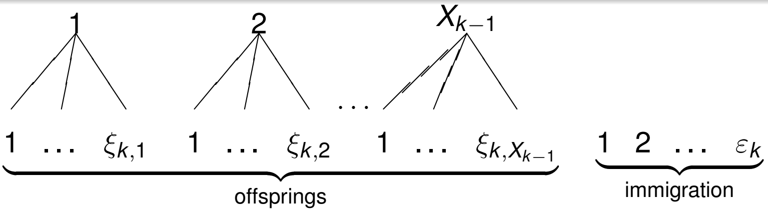
Parameters: $\alpha := \mathrm{E}\,\xi_{1,1}$, $\lambda := \mathrm{E}\,\varepsilon_1$, $b^2 := \mathrm{Var}\,\varepsilon_1$

Reformulation:

$$X_t = \alpha \circ X_{t-1} + \varepsilon_t$$

Classification:	$\alpha < 1$	$\alpha = 1$
	stable	**unstable**

Branching process with immigration (BPI)



$$X_k = \sum_{j=1}^{X_{k-1}} \xi_{k,j} + \varepsilon_k, \qquad X_0 = 0$$

$\{\xi_{k,j}, \varepsilon_k : j \in \mathbb{N}, k \in \mathbb{Z}_+\}$ independent

$\{\xi_{k,j} : j \in \mathbb{N}, k \in \mathbb{Z}_+\}$ identically distributed

$\{\varepsilon_k : k \in \mathbb{Z}_+\}$ identically distributed with $\mathrm{P}(\varepsilon_1 \neq 0) > 0$

Parameters: $m := \mathrm{E}\, \xi_{1,1}$, $\sigma^2 = \mathrm{Var}\, \xi_{1,1}$, $\lambda := \mathrm{E}\, \varepsilon_1$, $b^2 := \mathrm{Var}\, \varepsilon_1$

Classification:	$m < 1$	$m = 1$	$m > 1$
	subcritical	**critical**	**supercritical**

Conditional structure

Filtration: $\mathcal{F}_k := \sigma(X_0, X_1, \ldots, X_k), \quad k \in \mathbb{Z}_+$

Conditional expectation: $\mathsf{E}(X_k \mid \mathcal{F}_{k-1}) = mX_{k-1} + \lambda$

$M_k := X_k - \mathsf{E}(X_k \mid \mathcal{F}_{k-1}) = X_k - mX_{k-1} - \lambda, \quad k \in \mathbb{N}$

martingale differences, and we have

$$\boxed{X_k = \lambda + mX_{k-1} + M_k}$$

Conditional variance: $\mathsf{E}(M_k^2 \mid \mathcal{F}_{k-1}) = \sigma^2 X_{k-1} + b^2$

since

$$M_k = X_k - mX_{k-1} - \lambda = \sum_{j=1}^{X_{k-1}} (\xi_{k,j} - m) + (\varepsilon_k - \lambda)$$

Autoregressive process (AR)

AR(1) model

$$X_t = \mu + \alpha X_{t-1} + \varepsilon_t, \qquad t \in \mathbb{Z}$$

$\mu \in \mathbb{R}$ is the drift, $\alpha \in \mathbb{R}$ is the autoregressive parameter, and $\{\varepsilon_t,\ t \in \mathbb{Z}\}$ is a sequence of martingale differences

Classification:

$\alpha < 1$	$\alpha = 1$	$\alpha > 1$
stable	**unstable**	**explosive**

Connection

- All INAR(1) process is a branching process with immigration.
- All branching process with immigration is an AR(1) processes with drift and conditionally heteroscedasticity.

INAR(1) process with a seasonal structure

INAR(1)$_s$ model (Bourguignon, Vasconcellos, Reisen, I (2014))

$$Y_t = \sum_{j=1}^{Y_{t-s}} \xi_{t,j} + \varepsilon_t, \qquad t \in \mathbb{Z},$$

$\{\xi_{t,j} : t \in \mathbb{Z}, j \in \mathbb{N}\}$ and $\{\varepsilon_t : t \in \mathbb{Z}\}$ are independent, non-negative, integer-valued, identically distributed r.v.'s

$\mathrm{P}(\xi_{1,1} \in \{0, 1\}) = 1$, i.e., $\xi_{1,1}$ has Bernoulli distribution

$s \in \mathbb{N}$ denotes the seasonal period

Parameters: $\phi := \mathrm{E}\, \xi_{1,1}$, $\lambda := \mathrm{E}\, \varepsilon_1$

Reformulation:

$$Y_t = \phi \circ Y_{t-s} + \varepsilon_t$$

Classification:	$\phi < 1$	$\phi = 1$
	stable	**unstable**

Stationarity and second order properties

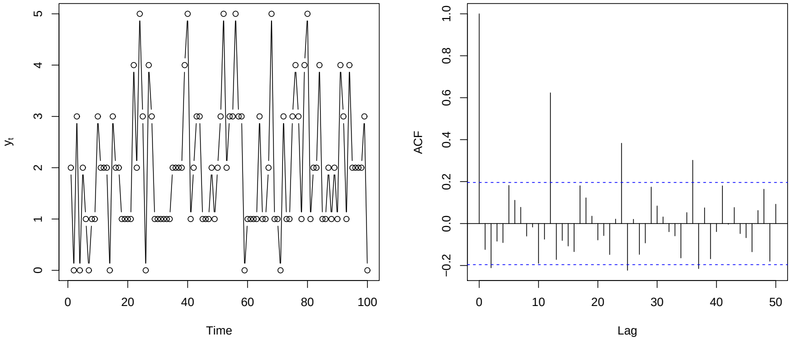
If $\phi \in [0,1)$, the unique stationary marginal distribution of INAR(1)$_s$ model can be expressed in terms of $\{\varepsilon_t : t \in \mathbb{Z}\}$ as

$$Y_t \stackrel{d}{=} \sum_{k=0}^{\infty} \phi^k \circ \varepsilon_{t-ks} = \varepsilon_t + \sum_{k=1}^{\infty} \sum_{j=1}^{\varepsilon_{t-sk}} Z_{t,k,j}, \quad t \in \mathbb{Z},$$

where $\stackrel{d}{=}$ stands for equality in distribution and $Z_{t,k,j} \sim \mathcal{B}e(\phi^k)$. Let $\{\varepsilon_t : t \in \mathbb{Z}\}$ be an i.i.d. sequence of Poisson distributed variables with mean $\lambda \in \mathbb{R}_+$ and let $\phi \in [0,1)$. Then the unique stationary solution satisfies $Y_t \sim \mathcal{P}o(\lambda/(1-\phi))$ and the autocorrelation function is given by

$$\rho(k) = \begin{cases} \phi^{k/s}, & \text{if } k \text{ is a multiple of } s, \\ 0, & \text{otherwise.} \end{cases}$$

100 simulated values of the INAR(1)$_s$ process and its sample autocorrelation function for $\phi = 0.5$, $\lambda = 1$ and $s = 12$.



Estimation methods: conditional least squares (CLS)

The conditional least squares estimator of $\theta = (\phi, \lambda)^\mathsf{T}$ is given by

$$\widehat{\theta}_{\mathrm{CLS}} := \arg\min_{\theta} \sum_{t=s+1}^{n} [Y_t - \mathbb{E}_\theta(Y_t|\mathcal{F}_{t-1})]^2$$

with $\mathbb{E}_\theta(Y_t|\mathcal{F}_{t-1}) = \mathbb{E}_\theta(Y_t|Y_{t-s}) = g(\theta, Y_{t-s})$, where $g(\theta, y) := \phi y + \lambda$. Solving the normal equations we have

$$\widehat{\phi}_{\mathrm{CLS}} := \frac{(n-s)\sum_{t=s+1}^{n} Y_t Y_{t-s} - \sum_{t=s+1}^{n} Y_t \sum_{t=s+1}^{n} Y_{t-s}}{(n-s)\sum_{t=s+1}^{n} Y_{t-s}^2 - \left(\sum_{t=s+1}^{n} Y_{t-s}\right)^2}$$

$$\widehat{\lambda}_{\mathrm{CLS}} := \frac{1}{n-s}\left(\sum_{t=s+1}^{n} Y_t - \widehat{\phi}_{\mathrm{CLS}} \sum_{t=s+1}^{n} Y_{t-s}\right)$$

Asymptotic result for conditional least squares

$$\sqrt{n}\begin{pmatrix}\widehat{\phi}_{\mathrm{CLS}} - \phi \\ \widehat{\lambda}_{\mathrm{CLS}} - \lambda\end{pmatrix} \xrightarrow{d} \mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix}, \Sigma\right)$$

where

$$\Sigma := \begin{bmatrix}\lambda^{-1}\phi(1-\phi)^2 + (1-\phi^2) & -(1+\phi)\lambda \\ -(1+\phi)\lambda & \lambda + (1+\phi)(1-\phi)^{-1}\lambda^2\end{bmatrix}$$

Estimation methods: conditional maximum likelihood (CML)

The INAR$(1)_s$ process consists of s mutually independent INAR(1) processes, thus it is an s-step Markov chain.

Hence, the conditional log-likelihood function is given by

$$\ell(\theta) = \log \mathbb{P}_\theta(Y_n, \ldots, Y_s | Y_{s-1}, \ldots, Y_0) = \sum_{t=s}^{n} \log[\mathbb{P}_\theta(Y_t | Y_{t-s})],$$

where

$$\mathbb{P}_\theta(Y_t | Y_{t-s}) = [\mathcal{B}i(Y_{t-s}, \phi) * \mathcal{P}o(\lambda)]\,(Y_t)$$
$$= e^{-\lambda} \sum_{i=0}^{\min(Y_t, Y_{t-s})} \frac{\lambda^{Y_t - i}}{(Y_t - i)!} \binom{Y_{t-s}}{i} \phi^i (1-\phi)^{Y_{t-s}-i}$$

Asymptotic result:

$$\sqrt{n} \begin{pmatrix} \widehat{\phi}_{\mathrm{CML}} - \phi \\ \widehat{\lambda}_{\mathrm{CML}} - \lambda \end{pmatrix} \xrightarrow{d} \mathcal{N}(0, \mathbf{I}^{-1}(\theta)),$$

where $\mathbf{I}(\theta)$ is a 2×2 Fisher information matrix.

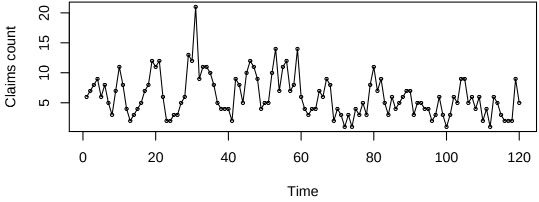
Monte Carlo simulation study

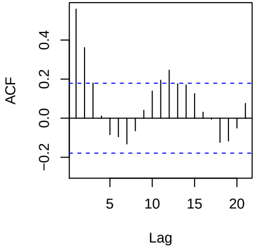
Table: Biases of estimators for $\lambda = 1$ (MSE in parenthesis)

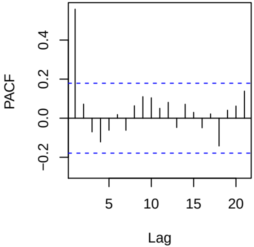
n	ϕ	Bias($\widehat{\phi}$)/MSE($\widehat{\phi}$) YW	CLS	CML	Bias($\widehat{\lambda}$)/MSE($\widehat{\lambda}$) YW	CLS	CML
	0.30	−0.0178	−0.0307	−0.0067	0.0315	0.0508	0.0040
		(0.0125)	(0.0133)	(0.0114)	(0.0353)	(0.0365)	(0.0291)
100	0.50	−0.0240	−0.0334	−0.0081	0.0500	0.0691	0.0064
		(0.0100)	(0.0116)	(0.0063)	(0.0549)	(0.0560)	(0.0304)
	0.80	−0.0267	−0.0362	−0.0031	0.1385	0.1854	0.0113
		(0.0058)	(0.0078)	(0.0012)	(0.1583)	(0.1921)	(0.0289)
	0.30	−0.0115	−0.0156	−0.0067	0.0221	0.0282	0.0115
		(0.0045)	(0.0044)	(0.0035)	(0.0130)	(0.0133)	(0.0104)
250	0.50	−0.0106	−0.0146	−0.0029	0.0254	0.0337	0.0057
		(0.0037)	(0.0040)	(0.0023)	(0.0174)	(0.0184)	(0.0109)
	0.80	−0.0143	−0.0166	−0.0016	0.0700	0.0823	0.0028
		(0.0019)	(0.0022)	(0.0004)	(0.0511)	(0.0572)	(0.0113)
	0.30	−0.0058	−0.0079	−0.0023	0.0063	0.0093	−0.0008
		(0.0022)	(0.0022)	(0.0018)	(0.0055)	(0.0056)	(0.0045)
500	0.50	−0.0033	−0.0056	−0.0007	0.0102	0.0148	0.0029
		(0.0018)	(0.0018)	(0.0010)	(0.0087)	(0.0089)	(0.0052)
	0.80	−0.0086	−0.0098	−0.0003	0.0468	0.0500	0.0043
		(0.0009)	(0.0009)	(0.0002)	(0.0237)	(0.0255)	(0.0055)

Real data example (Freeland)

Monthly counts of claims of short-term disability benefits reported to the Richmond, BC Workers Compensation Board.







Fitted models

Model		CML estimates	CLS estimates	AIC	BIC
$INAR(1)_{12}$	$\widehat{\phi}$	0.1746 (0.0036)	0.2410 (0.0899)	530.613	536.013
	$\widehat{\lambda}$	5.1391 (0.1951)	4.7554 (0.5897)		
INAR(1)	$\widehat{\phi}$	0.4418 (0.0029)	0.5510 (0.0783)	538.469	543.869
	$\widehat{\lambda}$	3.5224 (0.1364)	2.8526 (0.5079)		

The model fitted by CML estimation is

$$Y_t = 0.1746 \circ Y_{t-12} + \epsilon_t, \qquad \epsilon_t \sim \mathcal{P}o(5.1391)$$

INAR(1) process with serial and seasonal structure

Seasonal INAR({1,s}) model (I and Reisen (2014))

$$Z_t = \sum_{j=1}^{Z_{t-1}} \xi_{t,j} + \sum_{j=1}^{Z_{t-s}} \eta_{t,j} + \varepsilon_t, \qquad t \in \mathbb{Z},$$

$\{\xi_{t,j} : t \in \mathbb{Z}, j \in \mathbb{N}\}$, $\{\eta_{t,j} : t \in \mathbb{Z}, j \in \mathbb{N}\}$ and $\{\varepsilon_t : t \in \mathbb{Z}\}$ are independent, non-negative, integer-valued, i.d. r.v.'s

$\xi_{1,1}$ and $\eta_{1,1}$ have Bernoulli distribution

$s \in \mathbb{N}$ denotes the seasonal period

Parameters: $\alpha := \operatorname{E} \xi_{1,1}$, $\phi := \operatorname{E} \eta_{1,1}$, $\lambda := \operatorname{E} \varepsilon_1$

Reformulation:

$$Z_t = \alpha \circ Z_{t-1} + \phi \circ Z_{t-s} + \varepsilon_t$$

Classification:	$\alpha + \phi < 1$	$\alpha + \phi = 1$	$\alpha + \phi > 1$
	stable	**unstable**	**explosive**

State space representation

$$\boldsymbol{Z}_t = \boldsymbol{A} \circ \boldsymbol{Z}_{t-1} + \boldsymbol{\varepsilon}_t$$

where

$$\boldsymbol{A} := \begin{bmatrix} \alpha & 0 & \cdots & 0 & \phi \\ 1 & 0 & \cdots & 0 & 0 \\ & \ddots & & & \\ 0 & & \cdots & 1 & 0 \end{bmatrix} \qquad \boldsymbol{Z}_t := \begin{bmatrix} Z_t \\ Z_{t-1} \\ \vdots \\ Z_{t-s+1} \end{bmatrix} \qquad \boldsymbol{\varepsilon}_t := \begin{bmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The characteristic polynomial of $\boldsymbol{A}$ is given by

$$\det(x\boldsymbol{I} - \boldsymbol{A}) = x^s P(x^{-1})$$

where P denotes the autoregressive polynomial defined by

$$P(x) := 1 - \alpha x - \phi x^s$$

The INAR({1,s}) model is called primitive if the matrix $\boldsymbol{A}$ is primitive which holds iff $\alpha > 0$ and $\phi > 0$

Stationarity

Lemma

The roots of a primitive autoregressive polynomial P lie outside of the complex unit circle iff $\alpha + \phi < 1$. Then, for $|x| \leqslant 1$,

$$P(x)^{-1} = \sum_{j=0}^{\infty} \gamma_j x^j \quad \text{with} \quad \sum_{j=0}^{\infty} \gamma_j < \infty.$$

The non-negative sequence $\{\gamma_j : j \in \mathbb{Z}_+\}$ satisfies the recursion $\gamma_0 = 1$, $\gamma_j = \alpha\gamma_{j-1}$, $j = 1, \ldots, s-1$, $\gamma_j = \alpha\gamma_{j-1} + \phi\gamma_{j-s}$, $j \geq s$.

If $\alpha + \phi < 1$, the unique stationary marginal distribution of INAR({1,s}) model can be expressed in terms of $\{\varepsilon_t : t \in \mathbb{Z}\}$ as

$$Z_t \overset{d}{=} \sum_{k=0}^{\infty} \gamma_k \circ \varepsilon_{t-ks} = \varepsilon_t + \sum_{k=1}^{\infty} \sum_{j=1}^{\varepsilon_{t-sk}} U_{t,k,j}, \quad U_{t,k,j} \sim \mathcal{B}e(\gamma_k)$$

Second order properties

Let $\{\varepsilon_t : t \in \mathbb{Z}\}$ be an i.i.d. sequence of Poisson distributed variables with mean $\lambda \in \mathbb{R}_+$ and let $\phi \in [0, 1)$. Then the unique stationary solution satisfies $Y_t \sim \mathcal{P}o(\lambda/(1 - \alpha - \phi))$.

The autocorrelation function satisfies the recursion

$$\rho(k) = \alpha\rho(k-1) + \phi\rho(k-s), \qquad k \in \mathbb{Z}$$

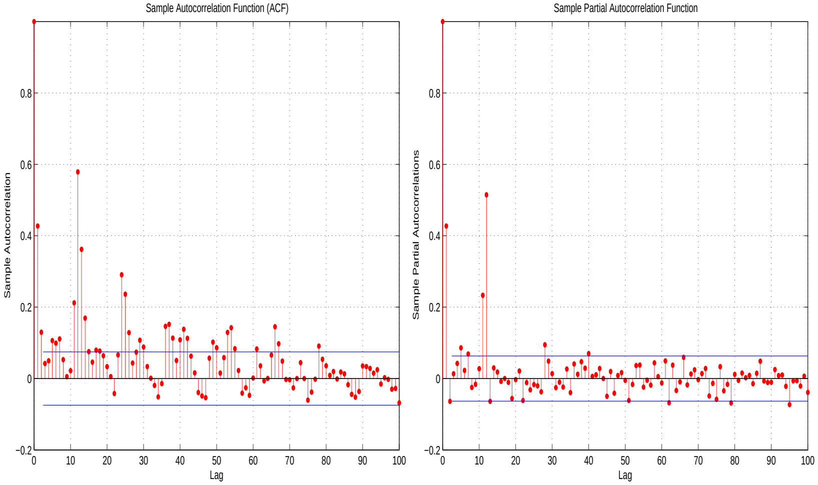
Recursive computation of the autocorrelation function starting from initial values $\rho(0) = 1$ and

$$\rho(k) = \alpha\rho(k-1) + \phi\rho(s-k), \qquad k = 1, \ldots, s-1$$

The partial autocorrelation function satisfies

$$\tau(k) \begin{cases} \neq 0, & \text{if } k = 0, 1, \ldots, s \\ = 0, & \text{otherwise.} \end{cases}$$

Sample ACF and PACF ($\alpha = 0.3$, $\phi = 0.5$ and $s = 12$)



Estimation methods: conditional least squares (CLS)

The conditional least squares estimator of $\theta = (\alpha, \phi, \lambda)^\top$ is given by

$$\widehat{\theta}_{\mathrm{CLS}} := \arg\min_{\theta} \sum_{t=s+1}^{n} [Y_t - \mathbb{E}_\theta(Y_t | \mathcal{F}_{t-1})]^2$$

with $\mathbb{E}_\theta(Y_t|\mathcal{F}_{t-1}) = \mathbb{E}_\theta(Y_t|Y_{t-1}, Y_{t-s}) = \alpha Y_{t-1} + \phi Y_{t-s} + \lambda$.

The normal equations are given by

$$\sum_{t=s+1}^{n} \begin{bmatrix} Y_{t-1} \\ Y_{t-s} \\ 1 \end{bmatrix} \begin{bmatrix} Y_{t-1} & Y_{t-s} & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \phi \\ \lambda \end{bmatrix} = \sum_{t=s+1}^{n} Y_t \begin{bmatrix} Y_{t-1} \\ Y_{t-s} \\ 1 \end{bmatrix}$$

Asymptotic result:

$$\sqrt{n}(\widehat{\theta}_{\mathrm{CLS}} - \theta) \xrightarrow{d} \mathcal{N}(0, \Sigma)$$

Estimation methods: conditional maximum likelihood (CML)

The conditional log-likelihood function is given by

$$\ell(\theta) = \log \mathbb{P}_\theta(Y_n, \ldots, Y_s | Y_{s-1}, \ldots, Y_0) = \sum_{t=s}^{n} \log[\mathbb{P}_\theta(Y_t | Y_{t-1}, Y_{t-s})],$$

where

$$\mathbb{P}_\theta(Y_t | Y_{t-1}, Y_{t-s}) = [\mathcal{B}i(Y_{t-1}, \alpha) * \mathcal{B}i(Y_{t-s}, \phi) * \mathcal{P}o(\lambda)]\,(Y_t)$$

Asymptotic result:

$$\sqrt{n} \begin{bmatrix} \widehat{\alpha}_{\mathrm{CML}} - \alpha \\ \widehat{\phi}_{\mathrm{CML}} - \phi \\ \widehat{\lambda}_{\mathrm{CML}} - \lambda \end{bmatrix} \xrightarrow{d} \mathcal{N}(0, \mathbf{I}^{-1}(\theta)),$$

where $\mathbf{I}(\theta)$ is a 3×3 Fisher information matrix.

Real data example revisited

The CLS estimates of parameters by solving the normal equations are

$$\widehat{\alpha} = 0.5388 \qquad \widehat{\phi} = 0.1561 \qquad \widehat{\lambda} = 1.8011$$

Thank you!

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