High Dimensional Minimum Risk Portfolio Optimization

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Outline

1. Motivation and Problem Statement
2. A Robust Approach to Minimum Risk Portfolio Optimization
3. Spiked Covariance Model in the Minimum Risk Portfolio Design
4. Concluding Remarks

Background

- Markowitz's Mean-Variance Portfolio Optimization Framework [Markowitz, 1952]

- Asset allocation: spread bets across multiple financial assets to minimize risk for given expected return, or maximize expected return for given risk
- Optimal solution specifies an "efficient frontier"

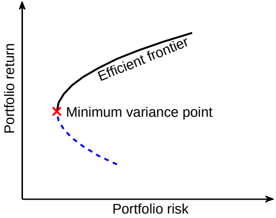

Figure: Efficient frontier

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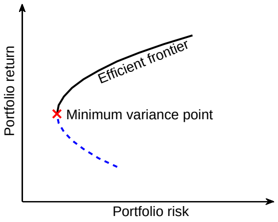

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  - Target: find the portfolio (lying on frontier) with minimal risk

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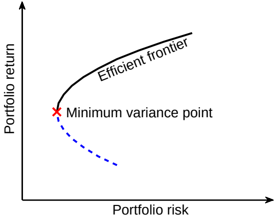

Figure: Efficient frontier

- Global Minimum Variance Portfolio Framework (GMVP)
  - Target: find the portfolio (lying on frontier) with minimal risk
  - Technical problem: require accurate covariance estimation

Problem Statement

- Asset allocation problem in GMVP framework

$$\min_{\mathbf{h}} \ \sigma^2(\mathbf{h}) = \mathbf{h}^T \mathbf{C}_N \mathbf{h}$$
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- Thus, in practice, GMVP selection is

$$\hat{\mathbf{h}}_{\text{GMVP}} = \frac{\hat{\mathbf{C}}_N^{-1}\mathbf{1}_N}{\mathbf{1}_N^T \hat{\mathbf{C}}_N^{-1}\mathbf{1}_N}$$

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$$\hat{\mathbf{C}}_{\mathrm{SCM}} = \frac{1}{n}\sum_{t=1}^{n} \tilde{\mathbf{x}}_t \tilde{\mathbf{x}}_t^T \qquad \text{with} \qquad \tilde{\mathbf{x}}_t = \mathbf{x}_t - \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i$$

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- **Robust shrinkage estimator of $\mathbf{C}_N$** [Yang et al., 2015]
- **Optimal shrinkage of eigenvalues in the spiked model**

Motivation and Objectives

- Robust covariance estimators [Tyler, 1987, Maronna, 1976]
  - For $n \gg N$, good performance for non-Gaussian samples; robust to outliers
  - For $n \approx O(N)$ performance degraded due to finite sampling
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- Shrinkage (regularized) robust estimators [Abramovich and Spencer, 2007, Pascal et al., 2013, Chen et al., 2011, Couillet and McKay, 2014]
  - Joint robustness and resilience to finite sampling
  - Key challenge: Design optimal shrinkage parameter for specified objective function
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  - Key challenge: Design optimal shrinkage parameter for specified objective function
    - Mean squared error minimization in [Couillet and McKay, 2014]
- **Objective:** Design shrinkage robust estimator for minimizing risk under the GMVP framework

System Model

- Data samples: a time series of independent and identically distributed observations $\mathbf{x}_t \in \mathbb{R}^N$ (returns of N assets)

$$\mathbf{x}_t = \boldsymbol{\mu} + \sqrt{\tau_t}\mathbf{C}_N^{1/2}\mathbf{y}_t, \quad t = 1, 2, ..., n$$

  - $\boldsymbol{\mu}$: mean vector of returns
  - $\mathbf{C}_N$: covariance matrix of returns
  - τ_t: real, positive scalar random variable, independent of $\mathbf{y}_t$
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- Embraces a class of elliptical distributions
  - Multivariate normal
  - Exponential
  - Multivariate Student-t
  - ...

Tyler's Robust M-estimator with Linear Shrinkage

- Shrinkage Tyler's estimator $\hat{\mathbf{C}}_{\mathrm{ST}}(\rho)$

  For $\rho \in (\max\{0, 1 - \frac{n}{N}\}, 1]$, the unique solution to

$$\hat{\mathbf{C}}_{\mathrm{ST}}(\rho) = (1-\rho)\frac{1}{n}\sum_{t=1}^{n}\frac{\tilde{\mathbf{x}}_t\tilde{\mathbf{x}}_t^T}{\frac{1}{N}\tilde{\mathbf{x}}_t^T\hat{\mathbf{C}}_{\mathrm{ST}}^{-1}(\rho)\tilde{\mathbf{x}}_t} + \rho\mathbf{I}_N$$

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- **Goal**: find the optimal ρ to minimize $\sigma^2(\hat{\mathbf{h}}_{\mathrm{ST}}(\rho))$
- **Main difficulty**: unknown $\mathbf{C}_N$

Method of Developing Risk-minimizing $\hat{\mathrm{C}}_{\mathrm{ST}}(\rho^o)$

- Step 1: Find the deterministic equivalent $N\bar{\sigma}^2(\rho)$ of $N\sigma^2(\hat{\mathbf{h}}_{\mathrm{ST}}(\rho))$ under double limits[1]
  - Non-random function of $\mathbf{C}_N$ and ρ
  - Gives deterministic approximation for the true portfolio risk

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- Step 3: Find the optimal ρ^o that minimizes $N\hat{\sigma}^2_{\mathrm{sc}}(\rho)$, and construct optimized portfolio based on this
  - $\hat{\mathbf{h}}^o_{\mathrm{ST}} = \frac{\hat{\mathbf{C}}^{o-1}_{\mathrm{ST}}(\rho^o)\mathbf{1}_N}{\mathbf{1}^T_N\hat{\mathbf{C}}^{o-1}_{\mathrm{ST}}(\rho^o)\mathbf{1}_N}$

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Deterministic Equivalent of Realized Portfolio Risk

- Recall realized portfolio risk: $\sigma^2(\hat{\mathbf{h}}_{\mathrm{ST}}(\rho)) = \frac{\mathbf{1}_N^T \hat{\mathbf{C}}_{\mathrm{ST}}^{-1}(\rho) \mathbf{C}_N \hat{\mathbf{C}}_{\mathrm{ST}}^{-1}(\rho) \mathbf{1}_N}{(\mathbf{1}_N^T \hat{\mathbf{C}}_{\mathrm{ST}}^{-1}(\rho) \mathbf{1}_N)^2}$

Theorem 1

Under mild assumptions,

$$\sup_{\rho \in \mathcal{R}_\varepsilon} \left| N\sigma^2(\hat{\mathbf{h}}_{\mathrm{ST}}(\rho)) - N\bar{\sigma}^2(\rho) \right| \xrightarrow{\text{a.s.}} 0$$

where

$$\bar{\sigma}^2(\rho) = \frac{1}{1 - \frac{\beta k^2}{(\gamma+\alpha k)^2}} \frac{\mathbf{1}_N^T \left(\frac{k}{(\gamma+\alpha k)} \mathbf{C}_N + \rho \mathbf{I}_N \right)^{-1} \mathbf{C}_N \left(\frac{k}{(\gamma+\alpha k)} \mathbf{C}_N + \rho \mathbf{I}_N \right)^{-1} \mathbf{1}_N}{\left(\mathbf{1}_N^T \left(\frac{k}{(\gamma+\alpha k)} \mathbf{C}_N + \rho \mathbf{I}_N \right)^{-1} \mathbf{1}_N \right)^2}$$

and where, for $\varepsilon \in (0, \min\{1, c^{-1}\})$, $\mathcal{R}_\varepsilon := [\varepsilon + \max\{0, 1 - c^{-1}\}, 1]$.

- $\kappa, \alpha, \beta, \gamma$ are functions of ρ and $\mathbf{C}_N$

Consistent Estimation of the Realized Portfolio Risk

- Recall realized portfolio risk: $$\sigma^2(\hat{\mathbf{h}}_{\mathrm{ST}}(\rho)) = \frac{\mathbf{1}_N^T \hat{\mathbf{C}}_{\mathrm{ST}}^{-1}(\rho)\mathbf{C}_N \hat{\mathbf{C}}_{\mathrm{ST}}^{-1}(\rho)\mathbf{1}_N}{(\mathbf{1}_N^T \hat{\mathbf{C}}_{\mathrm{ST}}^{-1}(\rho)\mathbf{1}_N)^2}$$

Theorem 2

Denote $\kappa = \lim_N \frac{1}{N}\mathrm{tr}[\mathbf{C}_N]$. *Under the settings of Theorem 1,*

$$\sup_{\rho\in\mathcal{R}_\varepsilon} \left| N\hat{\sigma}^2_{\mathrm{sc}}(\rho) - \frac{1}{\kappa}N\sigma^2(\hat{\mathbf{h}}_{\mathrm{ST}}(\rho)) \right| \xrightarrow{\mathrm{a.s.}} 0,$$

where

$$\hat{\sigma}^2_{\mathrm{sc}}(\rho) = \frac{(\hat{\gamma}_{\mathrm{sc}} + \hat{\alpha}_{\mathrm{sc}}\hat{k})^2}{\hat{k}\hat{\gamma}_{\mathrm{sc}}} \frac{\mathbf{1}_N^T \hat{\mathbf{C}}_{\mathrm{ST}}^{-1}(\rho)\left(\hat{\mathbf{C}}_{\mathrm{ST}}(\rho) - \rho\mathbf{I}_N\right)\hat{\mathbf{C}}_{\mathrm{ST}}^{-1}(\rho)\mathbf{1}_N}{(\mathbf{1}_N^T \hat{\mathbf{C}}_{\mathrm{ST}}^{-1}(\rho)\mathbf{1}_N)^2}.$$

- $\hat{k}, \hat{\gamma}_{\mathrm{sc}}, \hat{\alpha}_{\mathrm{sc}}$ are (observable) consistent estimators of $k, \gamma/\kappa, \alpha/\kappa$
- $\hat{\sigma}^2_{\mathrm{sc}}(\rho)$ only depends on observable quantities
- Since κ does not depend on ρ, minimizing $\sigma^2(\hat{\mathbf{h}}_{\mathrm{ST}}(\rho))$ over ρ is approximated by minimizing $\hat{\sigma}^2_{\mathrm{sc}}(\rho)$ over ρ

Consistent Estimation of the Realized Portfolio Risk

Corollary 3

Denote ρ^o and ρ^* the minimizers of $\hat{\sigma}^2_{\mathrm{sc}}(\rho)$ and $\sigma^2(\hat{\mathbf{h}}_{\mathrm{ST}}(\rho))$ over $\mathcal{R}_\varepsilon$ respectively. Under the settings of Theorem 1 and Theorem 2,

$$\left|N\sigma^2(\hat{\mathbf{h}}_{\mathrm{ST}}(\rho^o)) - N\sigma^2(\hat{\mathbf{h}}_{\mathrm{ST}}(\rho^*))\right| \xrightarrow{\mathrm{a.s.}} 0.$$

- In words....

  Choosing ρ to minimize $\hat{\sigma}^2_{\mathrm{sc}}(\rho)$ is as good (asymptotically) as minimizing the unobservable $\sigma^2(\hat{\mathbf{h}}_{\mathrm{ST}}(\rho))$

Portfolio Design for Minimizing Risk

- Find the optimal ρ via a numerical search

$$\rho^o = \underset{\rho \in [\varepsilon + \max\{0, 1 - c_N^{-1}\}, 1]}{\arg\min} \hat{\sigma}^2_{\mathrm{sc}}(\rho)$$

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- Construct the optimized portfolio:

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Synthetic Data Simulation

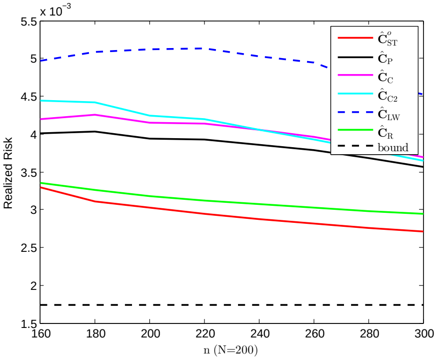


Figure: The average realized portfolio risk of different covariance estimators in the GMVP framework using synthetic data.

- Distribution of the asset returns: Student-T distribution (DoF=3)
- Benchmarks
  - $\hat{\mathbf{C}}_P$ Abramovich-Pascal Estimate [Couillet and McKay, 2014]
  - $\hat{\mathbf{C}}_C$ Chen Estimate [Couillet and McKay, 2014]
  - $\hat{\mathbf{C}}_{C2}$ [Chen et al., 2011]
  - $\hat{\mathbf{C}}_{\mathrm{LW}}$ [Ledoit and Wolf, 2004]
  - $\hat{\mathbf{C}}_{\mathrm{F}}$ [Rubio et al., 2012]
- $\hat{\mathbf{C}}^o_{\mathrm{ST}}$ achieves the smallest realized risk both when $n \leq N$ and $n > N$

Real Data Simulation

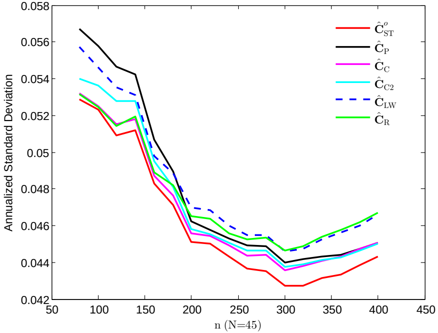
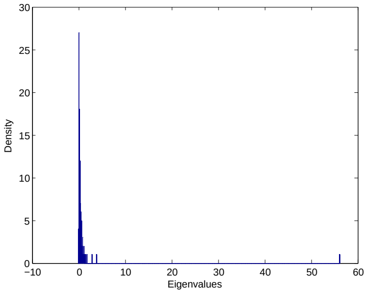


Figure: Realized portfolio risks achieved by different covariance estimators using HSI data set

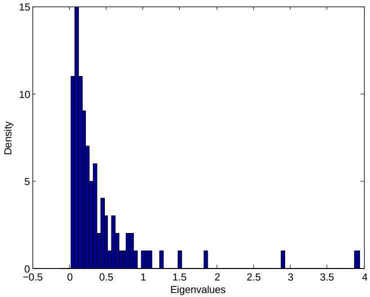
- 736 days of HSI daily returns (from Jan. 3, 2011 to Dec. 31, 2013)
- Rolling window method
- $\hat{\mathbf{C}}^o_{\mathrm{ST}}$ outperforms over the entire span of estimation windows
- Lack of stationarity when $n > 300$

Histogram of the Eigenvalue Distribution in Financial Data

Observe "spikes" in the spectrum distribution of real data



(a) All eigenvalues included



(b) The largest eigenvalue excluded

Figure: Histogram of eigenvalues of SCM of S&P100 data set.

System Model and Problem Formulation

- N-dimensional vectors $\mathbf{x}_t \overset{i.i.d.}{\sim} N(\mu, \mathbf{C}_N)$, $t = 1, ..., n$
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    - r: the number of spikes
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- Find the optimal $(w_1, ..., w_r)$ that minimize the portfolio risk:

$$\underset{w_1,...,w_r}{\arg\min}\ \sigma^2,$$

where $\sigma^2 = \frac{\mathbf{1}_N^T\hat{\mathbf{C}}_N^{-1}\mathbf{C}_N\hat{\mathbf{C}}_N^{-1}\mathbf{1}_N}{\left(\mathbf{1}_N^T\hat{\mathbf{C}}_N^{-1}\mathbf{1}_N\right)^2}$

Method of Developing Risk-minimizing $\hat{\mathbf{C}}_N^{-1}$

- Step 1: Find the deterministic equivalent $N\bar{\sigma}^2$ of $N\sigma^2$ under double limits[2]
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  - $\hat{\mathbf{h}}^* = \frac{\hat{\mathbf{C}}_{\text{risk}}^{-1}\mathbf{1}_N}{\mathbf{1}_N^T\hat{\mathbf{C}}_{\text{risk}}^{-1}\mathbf{1}_N}$

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Deterministic Equivalent of Realized Portfolio Risk

- Recall that $N\sigma^2 = \frac{\frac{1}{N}\mathbf{1}_N^T\hat{\mathbf{C}}_N^{-1}\mathbf{C}_N\hat{\mathbf{C}}_N^{-1}\mathbf{1}_N}{\left(\frac{1}{N}\mathbf{1}_N^T\hat{\mathbf{C}}_N^{-1}\mathbf{1}_N\right)^2}$

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- The numerator of $N\sigma^2$:

$$\frac{1}{N}\mathbf{1}_N^T\hat{\mathbf{C}}_N^{-1}\mathbf{C}_N\hat{\mathbf{C}}_N^{-1}\mathbf{1}_N = \frac{1}{N}\mathbf{1}_N^T(\mathbf{I}_N + w_1\mathbf{u}_1\mathbf{u}_1^T + ... + w_r\mathbf{u}_r\mathbf{u}_r^T)\times$$
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- The denominator of $N\sigma^2$:

$$\left(\frac{1}{N}\mathbf{1}_N^T \hat{\mathbf{C}}_N^{-1} \mathbf{1}_N\right)^2 = \left(\frac{1}{N}\mathbf{1}_N^T(\mathbf{I}_N + w_1\mathbf{u}_1\mathbf{u}_1^T + ... + w_r\mathbf{u}_r\mathbf{u}_r^T)\mathbf{1}_N\right)^2$$

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- Useful results: When $N, n \to \infty$,

$$\frac{1}{N}\mathbf{1}_N^T\mathbf{u}_i\mathbf{u}_i^T\mathbf{1}_N \xrightarrow{a.s.} s_i\frac{1}{N}\mathbf{1}_N^T\mathbf{v}_i\mathbf{v}_i^T\mathbf{1}_N, \quad \frac{1}{N}\mathbf{1}_N^T\mathbf{u}_i\mathbf{u}_i^T\mathbf{v}_i\mathbf{v}_i^T\mathbf{1}_N \xrightarrow{a.s.} s_i\frac{1}{N}\mathbf{1}_N^T\mathbf{v}_i\mathbf{v}_i^T\mathbf{1}_N$$
$$\frac{1}{N}\mathbf{1}_N^T\mathbf{u}_i\mathbf{u}_i^T\mathbf{v}_i\mathbf{v}_i^T\mathbf{u}_i\mathbf{u}_i^T\mathbf{1}_N \xrightarrow{a.s.} s_i^2\frac{1}{N}\mathbf{1}_N^T\mathbf{v}_i\mathbf{v}_i^T\mathbf{1}_N$$

where $s_i = \frac{1 - c/(t_i)^2}{1 + c/t_i}$, $i = 1, ..., r$.

Deterministic Equivalent of Realized Portfolio Risk

- Denote $k_i = \frac{1}{N}\mathbf{1}_N^T \mathbf{v}_i \mathbf{v}_i^T \mathbf{1}_N$. The deterministic equivalent of the numerator of $N\sigma^2$ is

$$(t_1 s_1^2 k_1 + s_1 k_1) w_1^2 + 2(s_1 k_1 + t_1 s_1 k_1) w_1 + \ldots$$
$$+ (t_r s_r^2 k_r + s_r k_r) w_r^2 + 2(s_r k_r + t_r s_r k_r) w_r + 1 + t_1 k_1 + \ldots + t_r k_r$$

- The deterministic equivalent of the denominator of $N\sigma^2$ is

$$(s_1 k_1 w_1 + s_2 k_2 w_2 + \ldots + s_r k_r w_r + 1)^2$$

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- The deterministic equivalent of the denominator of $N\sigma^2$ is

$$(s_1 k_1 w_1 + s_2 k_2 w_2 + \ldots + s_r k_r w_r + 1)^2$$

- Observe that $N\bar{\sigma}^2$ takes the form as

$$\frac{a_1 x_1^2 + a_2 x_2^2 + \ldots + a_r x_r^2 + f}{(b_1 x_1 + b_2 x_2 + \ldots + b_r x_r + d)^2} \tag{1}$$

We can use the Cauchy-Schwarz inequality to find $(x_1, x_2, \ldots, x_r)$ that minimize (1)

Optimal Shrinkage of Eigenvalues

- Since

$$(b_1x_1 + b_2x_2 + ... + b_rx_r + d)^2$$
$$\leq (a_1x_1^2 + a_2x_2^2 + ... + a_rx_r^2 + f)(\frac{b_1^2}{a_1} + \frac{b_2^2}{a_2} + ... + \frac{b_r^2}{a_r} + \frac{d^2}{f})$$

where "=" can be reached when $x_1 = \frac{b_1 f}{a_1 d}$, $x_2 = \frac{b_2 f}{a_2 d}$,...,$x_r = \frac{b_r f}{a_r d}$, we obtain that

$$\frac{a_1x_1^2 + a_2x_2^2 + ... + a_rx_r^2 + f}{(b_1x_1 + b_2x_2 + ... + b_rx_r + d)^2} \geq \frac{1}{\frac{b_1^2}{a_1} + \frac{b_2^2}{a_2} + ... + \frac{b_r^2}{a_r} + \frac{d^2}{f}}$$

- In our case,

$$f = 1 + t_1k_1 + ... + t_rk_r - \frac{(s_1k_1 + t_1s_1k_1)^2}{t_1s_1^2k_1 + s_1k_1} - ... - \frac{(s_rk_r + t_rs_rk_r)^2}{t_rs_r^2k_r + s_rk_r}$$

$$d = 1 - \frac{t_1s_1k_1 + s_1k_1}{t_1s_1 + 1} - ... - \frac{t_rs_rk_r + s_rk_r}{t_rs_r + 1}$$

$$a_1 = t_1s_1^2k_1 + s_1k_1 \qquad b_1 = s_1k_1$$

$$x_1 = w_1 + \frac{t_1 + 1}{t_1s_1 + 1}.$$

Precision Matrix Estimation

- The optimal $w_1^*(t_i, s_i, k_i) = \frac{b_1 f}{a_1 d} - \frac{t_1 + 1}{t_1 s_1 + 1}$, and similar results for $w_2^*, ..., w_r^*$
- The estimators of t_i, s_i and k_i, $i = 1, ..., r$:

$$\hat{t}_i = \frac{\lambda_i + 1 - c + \sqrt{(\lambda_i + 1 - c)^2 - 4\lambda_i}}{2} - 1$$

$$\hat{s}_i = \frac{1 - c/(\hat{t}_i)^2}{1 + c/\hat{t}_i}$$

$$\hat{k}_i = \frac{1}{\hat{s}_i}\frac{1}{N}\mathbf{1}_N^T\mathbf{u}_i\mathbf{u}_i^T\mathbf{1}_N$$

where $\lambda_1, ..., \lambda_r$ are the top r sample eigenvalues

- $\hat{\mathbf{C}}_{\text{risk}}^{-1} = \mathbf{I}_N + \hat{w}_1^*(\hat{t}_i, \hat{s}_i, \hat{k}_i)\mathbf{u}_1\mathbf{u}_1^T + ... + \hat{w}_r^*(\hat{t}_i, \hat{s}_i, \hat{k}_i)\mathbf{u}_r\mathbf{u}_r^T$

Numerical Simulations

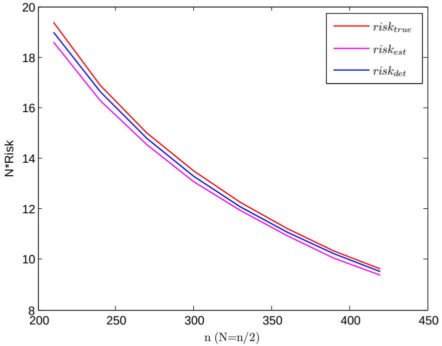


Figure: The deterministic equivalent and the estimator of the true risk

- $\mathbf{C}_N = \mathbf{I}_N + 14\mathbf{v}_1\mathbf{v}_1^T + 9\mathbf{v}_2\mathbf{v}_2^T + 4\mathbf{v}_3\mathbf{v}_3^T$
- $\mathbf{v}_1 = \sqrt{3/N}[\mathbf{1}_{N/3}; \mathbf{0}_{2N/3}]$
  $\mathbf{v}_2 = \sqrt{3/N}[\mathbf{1}_{N/3}; \mathbf{0}_{N/3}; \mathbf{1}_{N/3}]$
  $\mathbf{v}_3 = \sqrt{3/N}[\mathbf{0}_{2N/3}; \mathbf{1}_{N/3}]$

Numerical Simulations

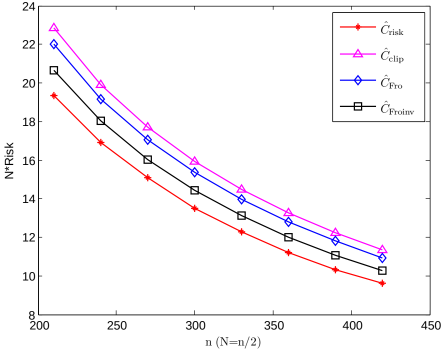


Figure: The average realized portfolio risk of different covariance estimators in the GMVP framework using synthetic data

- Benchmarks
  - $\hat{\mathbf{C}}_{\text{clip}}$: Eigenvalue Clipping [Laloux et al., 2000]
  - $\hat{\mathbf{C}}_{\text{Fro}}$: Frobenius norm minimization [Donoho et al., 2013]
  - $\hat{\mathbf{C}}_{\text{Froinv}}$: Frobenius norm minimization of precision matrix [Donoho et al., 2013]

Numerical Simulations

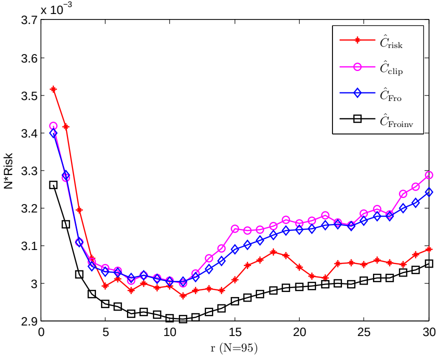
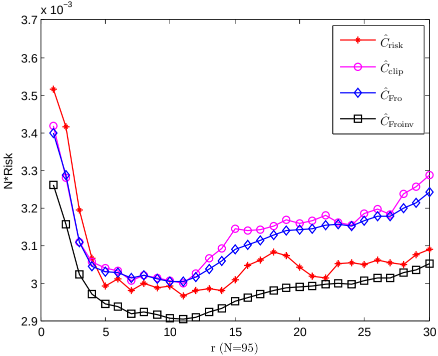


Figure: Realized portofio risks achieved out-of-sample over 1005 days of S&P100 real market data (from 2011 to 2014) under different number of assumed spikes

- 1005 days of daily returns of 95 stocks from S&P500 (from 2011 to 2014)
- Realized risk under different assumed number of spikes
- U-shape curve

Numerical Simulations



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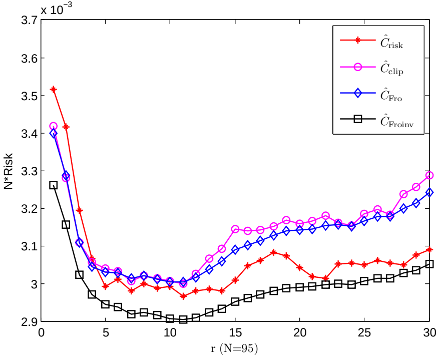

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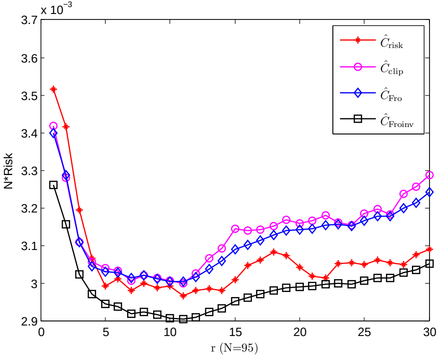

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Time Correlated Data Model

- $\mathbf{y}_t = \mathbf{x}_t \mathbf{T}^{1/2} \in \mathbb{R}^N$, $t = 1, ..., n$
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- New estimators $\hat{t}_i$,$\hat{s}_i$ and $\hat{k}_i$, $i = 1, ..., r$ [Vinogradova et al., 2013]
  - Define $\mathbf{Y}_N = [\mathbf{y}_1, ..., \mathbf{y}_n]$
  - Denote $\lambda_1 \geq ... \geq \lambda_N$ the eigenvalues of $\frac{1}{n}\mathbf{Y}_N \mathbf{Y}_N^T$
  - $\hat{m}(x) = \frac{1}{N-r} \sum_{j=r+1}^{N} \frac{1}{\lambda_j - x}$, $\hat{g}(x) = \hat{m}(x)(xc\hat{m}(x) + c - 1)$
  - $$\hat{t}_i = \left(\hat{g}(\lambda_i) \frac{1}{n} \mathrm{tr}[\frac{1}{N} \mathbf{Y}_N^T \mathbf{Y}_N] \right)^{-1}, \hat{s}_i = \frac{\hat{m}(\lambda_i)\hat{g}(\lambda_i)}{\hat{g}'(\lambda_i)}, \quad \hat{k}_i = \frac{1}{\hat{s}_i} \frac{1}{N} \mathbf{1}_N^T \mathbf{u}_i \mathbf{u}_i^T \mathbf{1}_N$$

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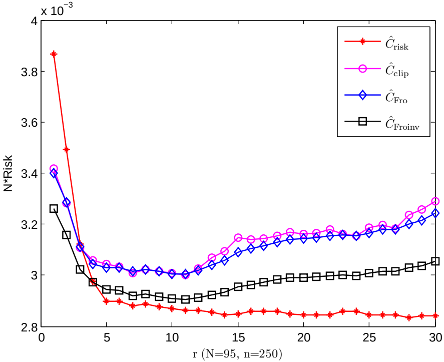
Numerical Simulations



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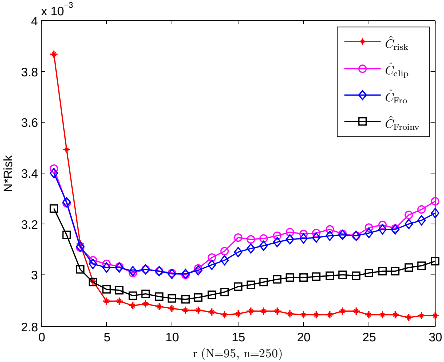

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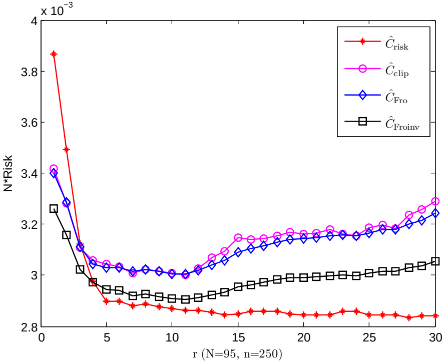

Figure: Realized portofio risks achieved out-of-sample over 1005 days of S&P100 real market data (from 2011 to 2014) under different number of assumed spikes

- $\hat{\mathbf{C}}_{\text{risk}}$ performs the best
- The risk realized by $\hat{\mathbf{C}}_{\text{risk}}$ doesn't change much when $r \geq 15$.

  Why?

  Haven't figured out yet...

Numerical Simulations

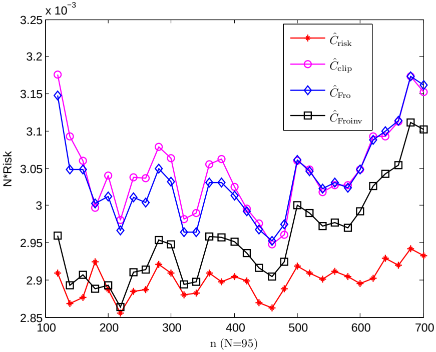

Figure: Realized portofio risks achieved out-of-sample over 1005 days of S&P100 real market data (from 2011 to 2014) under different number of samples

- $r = 11$

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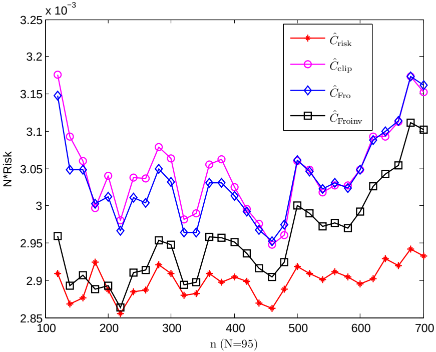

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- $r = 11$
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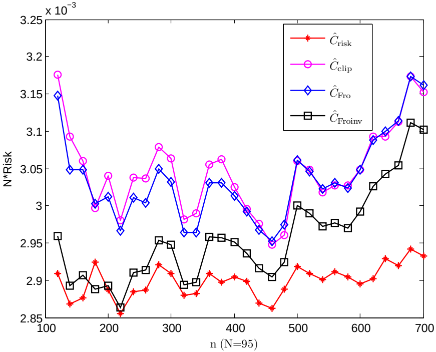

Figure: Realized portofio risks achieved out-of-sample over 1005 days of S&P100 real market data (from 2011 to 2014) under different number of samples

- $r = 11$
- Lack of stationarity when n grows too big
- Problem unsolved: the determination of the number of spikes

Concluding Remarks

- Two novel minimum risk portfolio optimization strategies
  - Shrinkage Tyler's robust M-estimator & Spiked covariance model
  - Deterministic characterization and consistent estimation of the portfolio risk
  - Parameter calibration to minimize risk

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  - Combine spiked model with robust estimation
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