Gegenbauer polynomials and positive definiteness

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Presentation of the problem

In Geostatistics one examines measurements depending on the location on the earth and on time. This leads to Random Fields of stochastic variables $Z(\xi, u)$ indexed by (ξ, u) belonging to $\mathbb{S}^2 \times \mathbb{R}$, where $\mathbb{S}^2$–the 2-dimensional sphere–is a model for the earth and $\mathbb{R}$ is a model for time.

If the variables are real-valued, one considers a basic probability space $(\Omega, \mathcal{F}, P)$, where all the random variables $Z(\xi, u)$ are defined as measurable mappings from Ω to $\mathbb{R}$.

The covariance of two stochastic variables X, Y is by definition

$$\mathrm{cov}(X, Y) = \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))).$$

For n variables $X_1, \ldots, X_n$ the covariance matrix

$$[\mathrm{cov}(X_k, X_l)]_{k,l=1}^{n}$$

is symmetric and positive semi-definite.

Isotropic and stationary covariance kernels

One is interested in isotropic and stationary random fields $Z(\xi, u),\ (\xi, u) \in \mathbb{S}^2 \times \mathbb{R}$, i.e., the situation where there exists a continuous function $f : [-1, 1] \times \mathbb{R} \to \mathbb{R}$ such that the covariance kernel is given as

$$\mathrm{cov}(Z(\xi, u), Z(\eta, v)) = f(\xi \cdot \eta, v - u), \quad \xi, \eta \in \mathbb{S}^2,\ u, v \in \mathbb{R}.$$

Here $\xi \cdot \eta = \cos(\theta(\xi, \eta))$ is the scalar product equal to cosine of the length of the geodesic arc (=angle) between ξ and η.
We require with other words that the covariance kernel only depends on the geodesic distance between the points on the sphere and on the time difference.

First main result

We shall characterize the class $\mathcal{P}(\mathbb{S}^2, \mathbb{R})$ of continuous functions $f : [-1, 1] \times \mathbb{R} \to \mathbb{R}$ which are positive definite in the following sense:
For any $n \in \mathbb{N}$ and any $(\xi_1, u_1), \ldots, (\xi_n, u_n) \in \mathbb{S}^2 \times \mathbb{R}$ the matrix

$$[f(\xi_k \cdot \xi_l, u_l - u_k)]_{k,l=1}^n$$

is symmetric and positive semi-definite.

Theorem (B-Porcu 2015)

The functions $f \in \mathcal{P}(\mathbb{S}^2, \mathbb{R})$ are precisely the functions

$$f(x, u) = \sum_{n=0}^{\infty} \varphi_n(u) P_n(x), \quad \sum_{n=0}^{\infty} \varphi_n(0) < \infty,$$

where (φ_n) is a sequence of real-valued continuous positive definite functions on $\mathbb{R}$ and P_n are the Legendre polynomials on $[-1, 1]$ normalized as $P_n(1) = 1$. The series is uniformly convergent.

Generalizations

This Theorem can be generalized in various ways:

- The sphere $\mathbb{S}^2$ can be replaced by $\mathbb{S}^d, d = 1, 2, \ldots$.
- The additive group $\mathbb{R}$ can be replaced by any locally compact group G.
- The sphere $\mathbb{S}^2$ can be replaced by the Hilbert sphere $\mathbb{S}^\infty$.

We shall characterize the set $\mathcal{P}(\mathbb{S}^d, G)$ of continuous functions $f : [-1, 1] \times G \to \mathbb{C}$ such that the kernel

$$f(\xi \cdot \eta, u^{-1}v)$$

is positive definite on $(\mathbb{S}^d \times G)^2$.
Here $d = 1, 2, \ldots, \infty$.
If $G = \{e\}$ is the trivial group we get a classical Theorem of Schoenberg from 1942 about positive definite functions on spheres.

To formulate these generalizations we need to recall:
The Gegenbauer polynomials $C_n^{(\lambda)}$ for $\lambda > 0$ are given by the generating function

$$(1 - 2xr + r^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(x) r^n, \quad |r| < 1, x \in \mathbb{C}. \qquad (1)$$

For $\lambda > 0$, we have the classical orthogonality relation:

$$\int_{-1}^{1} (1 - x^2)^{\lambda - 1/2} C_n^{(\lambda)}(x) C_m^{(\lambda)}(x) \, \mathrm{d}x = \frac{\pi \Gamma(n + 2\lambda) 2^{1-2\lambda}}{\Gamma^2(\lambda)(n + \lambda) n!} \delta_{m,n}. \quad (2)$$

For $\lambda = 0$ we use the generating function

$$\frac{1 - xr}{1 - 2xr + r^2} = \sum_{n=0}^{\infty} C_n^{(0)}(x) r^n, \quad |r| < 1, x \in \mathbb{C}. \tag{3}$$

It is well-known that

$$C_n^{(0)}(x) = T_n(x) = \cos(n \arccos x), n = 0, 1, \ldots$$

are the Chebyshev polynomials of the first kind.

$$\int_{-1}^{1} (1 - x^2)^{-1/2} T_n(x) T_m(x) \, \mathrm{d}x = \begin{cases} \frac{\pi}{2} \delta_{m,n} & \text{if } n > 0 \\ \pi \delta_{m,n} & \text{if } n = 0, \end{cases} \tag{4}$$

Warning: $\lambda \to C_n^{(\lambda)}(x)$ is discontinuous at $\lambda = 0$.

More on Gegenbauer polynomials

Putting $x = 1$ in the generating functions yields

$$C_n^{(\lambda)}(1) = (2\lambda)_n/n!, \quad \lambda > 0, \quad T_n(1) = 1.$$

Recall that for $a \in \mathbb{C}$

$$(a)_n = a(a+1)\cdots(a+n-1),\ n \geq 1, \quad (a)_0 = 1.$$

It is of fundamental importance that

$$|C_n^{(\lambda)}(x)| \leq C_n^{(\lambda)}(1), \quad x \in [-1,1], \quad \lambda \geq 0.$$

The special value $\lambda = (d-1)/2$ is relevant for the d-dimensional sphere

$$\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} \mid ||x|| = 1\},\ d \in \mathbb{N}$$

because of the relation of $C_n^{(d-1)/2}$ to spherical harmonics.

Spherical harmonics

A spherical harmonic of degree n for $\mathbb{S}^d$ is the restriction to $\mathbb{S}^d$ of a real-valued harmonic homogeneous polynomial in $\mathbb{R}^{d+1}$ of degree n.

$$\mathcal{H}_n(d) = \{\text{spherical harmonics of degree } n\} \subset \mathcal{C}(\mathbb{S}^d)$$

is a finite dimensional subspace of the continuous functions on $\mathbb{S}^d$. We have

$$N_n(d) := \dim \mathcal{H}_n(d) = \frac{(d)_{n-1}}{n!}(2n+d-1),\ n \geq 1, \quad N_0(d) = 1.$$

The surface measure of the sphere is denoted ω_d, and it is of total mass

$$||\omega_d|| = \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)}.$$

The spaces $\mathcal{H}_n(d)$ are mutual orthogonal subspaces of the Hilbert space $L^2(\mathbb{S}^d, \omega_d)$, which they generate.

Ultraspherical polynomials

This means that any $F \in L^2(\mathbb{S}^d, \omega_d)$ has an orthogonal expansion

$$F = \sum_{n=0}^{\infty} S_n,\ S_n \in \mathcal{H}_n(d), \quad ||F||_2^2 = \sum_{n=0}^{\infty} ||S_n||_2^2,$$

where the first series converges in $L^2(\mathbb{S}^d, \omega_d)$, and the second series is Parseval's equation. Here S_n is the orthogonal projection of F onto $\mathcal{H}_n(d)$ given as

$$S_n(\xi) = \frac{N_n(d)}{||\omega_d||} \int_{\mathbb{S}^d} c_n(d, \xi \cdot \eta) F(\eta)\, \mathrm{d}\omega_d(\eta),$$

where

$$c_n(d, x) = C_n^{((d-1)/2)}(x) / C_n^{((d-1)/2)}(1).$$

These polynomials are called ultraspherical polynomials or d-dimensional Legendre polynomials.
The 2-dimensional Legendre polynomials are the classical Legendre polynomials previously denoted P_n.

Orthogonality relation for ultraspherical polynomials

Specializing the orthogonality relation for the Gegenbauer polynomials to $\lambda = (d-1)/2$:

$$\int_{-1}^{1} (1-x^2)^{d/2-1} c_n(d,x) c_m(d,x)\,\mathrm{d}x = \frac{||\omega_d||}{||\omega_{d-1}||N_n(d)}\delta_{m,n}.$$

(Define $||\omega_0|| = 2$).
Note that

$$|c_n(d,x)| \le 1, \quad x \in [-1,1].$$

Schoenberg's Theorem from 1942

Let $\mathcal{P}(\mathbb{S}^d)$ denote the class of continuous functions $f : [-1,1] \to \mathbb{R}$ such that for any $n \in \mathbb{N}$ and for any $\xi_1, \ldots, \xi_n \in \mathbb{S}^d$ the $n \times n$ symmetric matrix

$$[f(\xi_k \cdot \xi_l)]_{k,l=1}^n$$

is positive semi-definite.

Theorem (Schoenberg 1942)

A function $f : [-1,1] \to \mathbb{R}$ belongs to the class $\mathcal{P}(\mathbb{S}^d)$ if and only if

$$f(x) = \sum_{n=0}^{\infty} b_{n,d} c_n(d,x), \quad x \in [-1,1],$$

for a non-negative summable sequence $(b_{n,d})_{n=0}^{\infty}$ given as

$$b_{n,d} = \frac{||\omega_{d-1}|| N_n(d)}{||\omega_d||} \int_{-1}^{1} f(x) c_n(d,x)(1-x^2)^{d/2-1}\, \mathrm{d}x.$$

Positive definite functions on groups

Consider an arbitrary locally compact group G, where we use the multiplicative notation, and in particular the neutral element of G is denoted e .
In the representation theory of these groups the following functions play an crucial role.
A continuous function $f : G \to \mathbb{C}$ is called positive definite if for any $n \in \mathbb{N}$ and any $u_1, \ldots, u_n \in G$ the $n \times n$-matrix

$$[f(u_k^{-1} u_l)]_{k,l=1}^n$$

is hermitian and positive semi-definite.
By $\mathcal{P}(G)$ we denote the set of continuous positive definite functions on G.

A generalization

We shall characterize the set $\mathcal{P}(\mathbb{S}^d, G)$ of continuous functions $f : [-1, 1] \times G \to \mathbb{C}$ such that the kernel

$$f(\xi \cdot \eta, u^{-1} v), \quad \xi, \eta \in \mathbb{S}^d,\ u, v \in G \tag{5}$$

is positive definite in the sense that for any $n \in \mathbb{N}$ and any $(\xi_1, u_1), \ldots (\xi_n, u_n) \in \mathbb{S}^d \times G$ the $n \times n$-matrix

$$\left[f(\xi_k \cdot \xi_l), u_k^{-1} u_l)\right]_{k,l=1}^{n} \tag{6}$$

is hermitian and positive semi-definite.
Note that for $G = \{e\}$ we can identify $\mathcal{P}(\mathbb{S}^d, G)$ with $\mathcal{P}(\mathbb{S}_d)$.

Simple properties

Proposition

(i) *For* $f_1, f_2 \in \mathcal{P}(\mathbb{S}^d, G)$ *and* $r \geq 0$ *we have* $rf_1, f_1 + f_2$*, and* $f_1 \cdot f_2 \in \mathcal{P}(\mathbb{S}^d, G)$*.*

(ii) *For a net of functions* $(f_i)_{i\in I}$ *from* $\mathcal{P}(\mathbb{S}^d, G)$ *converging pointwise to a continuous function* $f : [-1, 1] \times G \to \mathbb{C}$*, we have* $f \in \mathcal{P}(\mathbb{S}^d, G)$*.*

(iii) *For* $f \in \mathcal{P}(\mathbb{S}^d, G)$ *we have* $f(\cdot, e) \in \mathcal{P}(\mathbb{S}_d)$ *and* $f(1, \cdot) \in \mathcal{P}(G)$*.*

(iv) *For* $f \in \mathcal{P}(\mathbb{S}_d)$ *and* $g \in \mathcal{P}(G)$ *we have* $f \otimes g \in \mathcal{P}(\mathbb{S}^d, G)$*, where* $f \otimes g(x, u) := f(x)g(u)$ *for* $(x, u) \in [-1, 1] \times G$*. In particular we have* $f \otimes 1_G \in \mathcal{P}(\mathbb{S}^d, G)$ *and* $f \mapsto f \otimes 1_G$ *is an embedding of* $\mathcal{P}(\mathbb{S}_d)$ *into* $\mathcal{P}(\mathbb{S}^d, G)$*.*

Characterization of the class $\mathcal{P}(\mathbb{S}^d, G)$

Theorem (B-Porcu 2015)

Let $d \in \mathbb{N}$ and let $f : [-1,1] \times G \to \mathbb{C}$ be a continuous function. Then f belongs to $\mathcal{P}(\mathbb{S}^d, G)$ if and only if there exists a sequence $\varphi_{n,d} \in \mathcal{P}(G)$ with $\sum \varphi_{n,d}(e) < \infty$ such that

$$f(x,u) = \sum_{n=0}^{\infty} \varphi_{n,d}(u) c_n(d,x),$$

and the above expansion is uniformly convergent for $(x,u) \in [-1,1] \times G$. We have

$$\varphi_{n,d}(u) = \frac{N_n(d)||\omega_{d-1}||}{||\omega_d||} \int_{-1}^{1} f(x,u) c_n(d,x)(1-x^2)^{d/2-1}\,\mathrm{d}x.$$

Relation between the classes $\mathcal{P}(\mathbb{S}^d, G), d = 1, 2, \ldots$

Note that

$$\mathcal{P}(\mathbb{S}^1, G) \supset \mathcal{P}(\mathbb{S}^2, G) \supset \cdots$$

The inclusion $\mathcal{P}(\mathbb{S}^d, G) \subseteq \mathcal{P}(\mathbb{S}^{d-1}, G)$ is easy, since $\mathbb{S}^{d-1}$ can be considered as the equator of $\mathbb{S}^d$. That the inclusion is strict is more subtle.

The intersection

$$\bigcap_{d=1}^{\infty} \mathcal{P}(\mathbb{S}^d, G)$$

can be identified with the set $\mathcal{P}(\mathbb{S}^\infty, G)$ of continuous functions $f : [-1, 1] \times G \to \mathbb{C}$ such that for all n

$$[f(\xi_k \cdot \xi_l, u_k^{-1} v_l)]_{k,l=1}^n$$

is hermitean and positive semi-definite for $(\xi_k, u_k), k = 1, \ldots, n$ from $\mathbb{S}^\infty \times G$, where

$$\mathbb{S}^\infty = \{(x_k) \mid \sum_{k=1}^{\infty} x_k^2 = 1\} \subset \ell^2.$$

Schoenberg's second Theorem

When $G = \{e\}$ Schoenberg proved in 1942:

Theorem

A function $f : [-1,1] \to \mathbb{R}$ belongs to $\mathcal{P}(\mathbb{S}^\infty) = \cap_{d=1}^{\infty} \mathcal{P}(\mathbb{S}^d)$ if and only if

$$f(x) = \sum_{n=0}^{\infty} b_n x^n$$

for a non-negative summable sequence b_n. The convergence is uniform on $[-1,1]$.

A characterization of $\mathcal{P}(\mathbb{S}^\infty, G)$

Theorem (B-Porcu 2015)

Let G denote a locally compact group and let $f : [-1,1] \times G \to \mathbb{C}$ be a continuous function. Then f belongs to $\mathcal{P}(\mathbb{S}^\infty, G)$ if and only if there exists a sequence $\varphi_n \in \mathcal{P}(G)$ with $\sum \varphi_n(e) < \infty$ such that

$$f(x,u) = \sum_{n=0}^{\infty} \varphi_n(u) x^n,$$

and the above expansion is uniformly convergent for $(x,u) \in [-1,1] \times G$.

Schoenberg coefficient functions

For $f \in \mathcal{P}(\mathbb{S}^d, G)$ we know that also $f \in \mathcal{P}(\mathbb{S}^k, G)$ for $k = 1, 2, \ldots, d$ and therefore we have d expansions

$$f(x,u) = \sum_{n=0}^{\infty} \varphi_{n,k}(u) c_n(k,x), \quad (x,u) \in [-1,1] \times G,$$

where $k = 1, 2, \ldots, d$.
We call $\varphi_{n,k}(u)$ the *k-Schoenberg coefficient functions* of $f \in \mathcal{P}(\mathbb{S}^d, G)$.
In the case where $G = \{e\}$ the k-Schoenberg coefficient functions are non-negative constants and they are just called *k-Schoenberg coefficients*.

More about Schoenberg coefficient functions

There is a simple relation between these k-Schoenberg coefficients / coefficient functions:

Suppose $f \in \mathcal{P}(\mathbb{S}^{d+2}, G) \subset \mathcal{P}(\mathbb{S}^d, G)$. Then for $u \in G, n \geq 0$

$$\varphi_{n,d+2}(u) = \frac{(n+d-1)(n+d)}{d(2n+d-1)}\varphi_{n,d}(u) - \frac{(n+1)(n+2)}{d(2n+d+3)}\varphi_{n+2,d}(u).$$

In the case $G = \{e\}$ these relations were found by Gneiting (2013) and extended to general G in B-Porcu(2015) (available on the ArXive).

A result of J. Ziegel, 2014, and its generalization

Theorem (Johanna Ziegel, 2014)

Let $d \in \mathbb{N}$ and suppose that $f \in \mathcal{P}(\mathbb{S}^{d+2})$. Then f is continuously differentiable in the open interval $(-1, 1)$ and there exist $f_1, f_2 \in \mathcal{P}(\mathbb{S}^d)$ such that

$$f'(x) = \frac{f_1(x) - f_2(x)}{1 - x^2}, \quad -1 < x < 1.$$

Theorem (B-Porcu 2015)

Let $d \in \mathbb{N}$ and suppose that $f \in \mathcal{P}(\mathbb{S}^{d+2}, G)$. Then $f(x, u)$ is continuously differentiable with respect to x in $]-1, 1[$ and

$$\frac{\partial f(x, u)}{\partial x} = \frac{f_1(x, u) - f_2(x, u)}{1 - x^2}, \quad (x, u) \in]-1, 1[\times G$$

for functions $f_1, f_2 \in \mathcal{P}(\mathbb{S}^d, G)$. In particular $\frac{\partial f(x,u)}{\partial x}$ is continuous on $]-1, 1[\times G$.

Some remarks

The result of Ziegel is the analogue of an old result of Schoenberg about radial positive definite functions (Ann. Math. 1938):
If $f : [0,\infty[\to \mathbb{R}$ is a continuous functions such that $f(||x||)$ is positive definite on $\mathbb{R}^n$, then f has a continuous derivative of order $[(n-1)/2]$ on $(0,\infty)$.
The above extension of Ziegel's result to general G is needed in our extension of Schoenberg's result to $\mathcal{P}(\mathbb{S}^\infty, G)$.
If $f \in \mathcal{P}(\mathbb{S}^\infty, G)$ we have a sequence of expansions, $d = 1, 2, \ldots$

$$f(x,u) = \sum_{n=0}^{\infty} \varphi_{n,d}(u) c_n(d,x) = \sum_{n=0}^{\infty} \varphi_n(u) x^n$$

valid for $(x,u) \in [-1,1] \times G$.
Here $\varphi_{n,d}(u), \varphi_n(u) \in \mathcal{P}(G)$. As part of our proof we obtain

$$\lim_{d\to\infty} \varphi_{n,d}(u) = \varphi_n(u)$$

for each $n \in \mathbb{N}_0, u \in G$.

Some indications of proof

Lemma

Any $f \in \mathcal{P}(\mathbb{S}^d, G)$ *satisfies*

$$f(x, u^{-1}) = \overline{f(x, u)}, \quad |f(x, u)| \leq f(1, e), \quad (x, u) \in [-1, 1] \times G.$$

Lemma

Let $K \subset G$ *be a non-empty compact set, and let* $\delta > 0$ *and an open neighbourhood* U *of* $e \in G$ *be given. Then there exists a partition of* $\mathbb{S}^d \times K$ *in finitely many non-empty disjoint Borel sets, say* $M_j, j = 1, \ldots, r$*, such that each* M_j *has the property*

$$(\xi, u), (\eta, v) \in M_j \implies \theta(\xi, \eta) < \delta, \; u^{-1}v \in U.$$

The crucial Lemma

Lemma

For a continuous function $f : [-1,1] \times G \to \mathbb{C}$ the following are equivalent:

(i) $f \in \mathcal{P}(\mathbb{S}^d, G)$.

(ii) *f is bounded and for any complex Radon measure μ on $\mathbb{S}^d \times G$ of compact support we have*

$$\int_{\mathbb{S}^d \times G} \int_{\mathbb{S}^d \times G} f(\cos\theta(\xi,\eta), u^{-1}v)\, \mathrm{d}\mu(\xi,u)\, \mathrm{d}\overline{\mu(\eta,v)} \geq 0.$$

Having established the Lemma, the next idea is to apply (ii) to the measure $\mu = \omega_d \otimes \sigma$, where σ is an arbitrary complex Radon measure of compact support on G.

Application to some homogeneous spaces

In a recent manuscript Guella, Menegatto and Peron prove characterization results for isotropic positive definite kernels on $\mathbb{S}^d \times \mathbb{S}^{d'}$ for $d, d' \in \mathbb{N} \cup \{\infty\}$. They consider the set $\mathcal{P}(\mathbb{S}^d, \mathbb{S}^{d'})$ of continuous functions $f : [-1,1]^2 \to \mathbb{R}$ such that the kernel

$$K((\xi,\zeta),(\eta,\chi) = f(\xi\cdot\eta,\zeta\cdot\chi), \quad \xi,\eta \in \mathbb{S}^d, \quad \zeta,\chi \in \mathbb{S}^{d'},$$

is positive definite in the sense that for any $n \in \mathbb{N}$ and any $(\xi_1,\zeta_1),\ldots,(\xi_n,\zeta_n) \in \mathbb{S}^d \times \mathbb{S}^{d'}$ the matrix

$$[K((\xi_k,\zeta_k),(\xi_l,\zeta_l))]_{k,l=1}^n$$

is positive semi-definite.
They prove the following:

Theorem of Gyuella, Menegatto, Peron, 2015

Theorem

Let $d, d' \in \mathbb{N}$ and let $f : [-1,1]^2 \to \mathbb{R}$ be a continuous function. Then $f \in \mathcal{P}(\mathbb{S}^d, \mathbb{S}^{d'})$ if and only if

$$f(x,y) = \sum_{n,m=0}^{\infty} \widehat{f}_{n,m} c_n(d,x) c_m(d',y), \quad x, y \in [-1,1],$$

where $\widehat{f}_{n,m} \geq 0$ such that $\sum \widehat{f}_{n,m} < \infty$. The above expansion is uniformly convergent, and we have

$$\widehat{f}_{n,m} = \frac{N_n(d)\sigma_{d-1}}{\sigma_d} \frac{N_m(d')\sigma_{d'-1}}{\sigma_{d'}} \times$$
$$\int_{-1}^{1} \int_{-1}^{1} f(x,y) c_n(d,x) c_m(d',y) (1-x^2)^{d/2-1} (1-y^2)^{d'/2-1} \, \mathrm{d}x \, \mathrm{d}y.$$

Reduction to our results

The idea of proof is to consider $\mathbb{S}^{d'}$ as the homogeneous space $O(d'+1)/O(d')$, where $O(d'+1)$ is the compact group of orthogonal transformations in $\mathbb{R}^{d'+1}$ and $O(d')$ is identified with the subgroup of $O(d'+1)$ which fixes the point $\varepsilon_1 = (1, 0, \dots, 0) \in \mathbb{S}^{d'+1}$.
It is elementary to see that the formula of the Theorem defines a function $f \in \mathcal{P}(\mathbb{S}^d, \mathbb{S}^{d'})$.
Let us next consider $f \in \mathcal{P}(\mathbb{S}^d, \mathbb{S}^{d'})$ and define $F : [-1,1] \times O(d'+1) \to \mathbb{R}$ by

$$F(x, A) = f(x, A\varepsilon_1 \cdot \varepsilon_1), \quad x \in [-1,1],\ A \in O(d'+1).$$

Then $F \in \mathcal{P}(\mathbb{S}^d, O(d'+1))$ because

$$F(x, B^{-1}A) = f(x, A\varepsilon_1 \cdot B\varepsilon_1), \quad A, B \in O(d'+1).$$

Continuation of the proof 1

By our main Theorem

$$F(x,A)=\sum_{n=0}^{\infty}\varphi_{n,d}(A)c_n(d,x),\quad x\in[-1,1],\ A\in O(d'+1),$$

and

$$\varphi_{n,d}(A)=\frac{N_n(d)\sigma_{d-1}}{\sigma_d}\int_{-1}^{1}f(x,A\varepsilon_1\cdot\varepsilon_1)c_n(d,x)(1-x^2)^{d/2-1}\,\mathrm{d}x$$

belongs to $\mathcal{P}(O(d'+1))$.
The function $\varphi_{n,d}$ is bi-invariant under $O(d')$, i.e.,

$$\varphi_{n,d}(KAL)=\varphi_{n,d}(A),\quad A\in O(d'+1),\, K,L\in O(d').$$

This is simply because $f(x,KAL\varepsilon_1\cdot\varepsilon_1)=f(x,A\varepsilon_1\cdot\varepsilon_1)$.

Continuation of the proof 2

The mapping $A \mapsto A\varepsilon_1$ is a continuous surjection of $O(d'+1)$ onto $\mathbb{S}^{d'}$, and it induces a homeomorphism of the homogeneous space $O(d'+1)/O(d')$ onto $\mathbb{S}^{d'}$.
It is easy to see that as a bi-invariant function, $\varphi_{n,d}$ has the form

$$\varphi_{n,d}(A) = g_{n,d}(A\varepsilon_1 \cdot \varepsilon_1)$$

for a uniquely determined continuous function $g_{n,d} : [-1,1] \to \mathbb{R}$.
We have in addition $g_{n,d} \in \mathcal{P}(\mathbb{S}^{d'})$, because for $\xi_1, \ldots, \xi_n \in \mathbb{S}^{d'}$ there exist $A_1, \ldots, A_n \in O(d'+1)$ such that
$\xi_j = A_j\varepsilon_1, j = 1, \ldots, n$, hence

$$g_{n,d}(\xi_k \cdot \xi_l) = g_{n,d}(A_l^{-1}A_k\varepsilon_1 \cdot \varepsilon_1) = \varphi_{n,d}(A_l^{-1}A_k).$$

It is now easy to finish the proof.

Some references

C. Berg, E. Porcu, *From Schoenberg coefficients to Schoenberg functions*, Preprint submitted to ArXiv.

J. C. Guella, V. A. Menegatto and A. P. Peron, *An extension of a theorem of Schoenberg to products of spheres* arXiv:1503.08174.

Thank you for your attention