Modularity, Synchronization, and What Robotics May Yet Learn from the Brain

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1 Introduction

I am truly delighted to contribute to this book on the occasion of Professor Suguru's Arimoto's 70th birthday. Professor Arimoto's work is probably the one I most admire in the whole field of robotics. Like the founders of cybernetics [86, 87, 93, 3], Professor Arimoto was also deeply interested in neuroscience from the start, and actually he was one of the main contributors to the now very classical FitzHugh-Nagumo [11, 55] model of neural oscillators. So I believe it is fitting to write an article about what robotics may yet learn from the brain in the context of this celebration.

Although neurons as computational elements are 7 orders of magnitude slower than their artificial counterparts, the primate brain grossly outperforms robotic algorithms in all but the most structured tasks. Parallelism alone is a poor explanation, and much recent functional modelling of the central nervous system focuses on its modular, heavily feedback-based computational architecture, the result of accumulation of subsystems throughout evolution. We have extensively discussed in some earlier work [76, 74, 44] this architecture from a global stability and convergence point of view. In this article, we describe recent work [88] on synchronization as a model of computations at different scales in the brain, such as pattern matching, temporal binding of sensory data, and mirror neuron response. Finally, we derive [63] a simple condition for a general dynamical system to globally converge to a regime where multiple groups of fully synchronized elements coexist. Applications of such "polyrhythms" to some classical questions in robotics and systems neuroscience are discussed.

The development makes extensive use of nonlinear contraction theory, a comparatively recent analysis tool whose main features will be briefly reviewed. In particular,

- Global results on synchronization can be obtained using most common models of neural oscillators, such as the FitzHugh-Nagumo model.

- Since contraction is preserved under most common system combinations (parallel, hierarchies, negative feedback), it represents a natural framework for motor primitives.
- In locomotion, the analysis exhibits none of the topological difficulties that may arise when coupling large numbers of phase oscillators, and it guarantees global exponential convergence.
- Replacing ordinary CPG connections by filters enables automatic frequency-based gate selection.
- Stable polyrhythmic aggregates of arbitrary size can be constructed recursively, motivated by evolution and developmemt.
- Just as global synchronization occurs naturally and quantifiably in networks of locally coupled oscillators, it can be turned off by adding a single inhibitory connection.
- In vision and pattern recognition, detectors for various types of symmetries can be systematically constructed.

2 Modularity, Stability, and Evolution

Basically, a nonlinear time-varying dynamic system will be called *contracting* if initial conditions or temporary disturbances are forgotten exponentially fast, i.e., if trajectories of the perturbed system return to their nominal behavior with an exponential convergence rate. It turns out that relatively simple conditions can be given for this stability-like property to be verified, and furthermore that this property is preserved through basic system combinations (such as series, hierarchies, parallel combinations, and negative feedback), a state-space feature reminiscent in spririt of input-output passivity [65]. Furthermore, as discussed in section 3, the concept of *partial contraction* allows to extend the applications of contraction analysis to include convergence to behaviors or to specific properties (such as equality of state components, or convergence to a manifold) rather than trajectories. This section is largely based on [43, 76, 74, 44] to which the reader is referred for details.

We consider general time-varying deterministic systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1}$$

where $\mathbf{f}$ is an $n \times 1$ nonlinear vector function and $\mathbf{x}$ is the $n \times 1$ state vector. The above equation may also represent the closed-loop dynamics of a controlled system with state feedback $\mathbf{u}(\mathbf{x}, t)$. All quantities are assumed to be real and smooth, by which it is meant that any required derivative or partial derivative exists and is continuous. The basic result of [43] can then be stated as

Theorem 1. *Consider system (1), and assume there exists a uniformly positive definite metric*

$$\mathbf{M}(\mathbf{x}, t) \ = \ \mathbf{\Theta}'(\mathbf{x}, t) \ \mathbf{\Theta}(\mathbf{x}, t)$$

such that the associated generalized Jacobian

$$\mathbf{F} \ = \ \left(\dot{\mathbf{\Theta}} + \mathbf{\Theta} \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right) \mathbf{\Theta}^{-1}$$

is uniformly negative definite. Then all system trajectories converge exponentially to a single trajectory, with convergence rate $|\lambda_{max}|$*, where* λ_{max} *is the largest eigenvalue of the symmetric part of* $\mathbf{F}$*. The system is said to be contracting.*

By $\mathbf{\Theta}'$ we mean the Hermitian (conjugate transpose) of $\mathbf{\Theta}$, and by symmetric part of $\mathbf{F}$ we mean $\frac{1}{2}(\mathbf{F} + \mathbf{F}')$. It can be shown conversely that the existence of a uniformly positive definite metric with respect to which the system is contracting is also a necessary condition for global exponential convergence of trajectories. In the linear time-invariant case, a system is globally contracting if and only if it is strictly stable, with $\mathbf{F}$ simply being a normal Jordan form of the system and $\mathbf{\Theta}$ the coordinate transformation to that form. The results immediately extend to the case where the state is in $\mathbb{C}^n$.

Example [74] Consider the Lorenz system

$$\begin{aligned} \dot{x} &= \sigma \ (y - x) \\ \dot{y} &= \rho \, x \ - \ y \ - \ x \, z \\ \dot{z} &= - \ \beta \ z \ + \ x \ y \end{aligned}$$

with strictly positive constants σ, ρ, β, and, given measurements of x, the reduced-order identity observer [62, 58]

$$\begin{aligned} \dot{\hat{y}} &= \rho \, x \ - \ \hat{y} \ - \ x \, \hat{z} \\ \dot{\hat{z}} &= - \ \beta \ \hat{z} \ + \ x \ \hat{y} \end{aligned}$$

The symmetric part of the observer's Jacobian is $-\ \mathrm{diag}(1, \beta)$, and thus the observer is contracting with an identity metric. Since by construction $(\hat{y}, \hat{z}) = (y, z)$ is a particular solution, the estimated state converges exponentially to the actual state, with rate $\min(1, \beta)$.

An important property is that, under mild conditions, contraction is preserved through system combinations such as parallel, series or hierarchies, translation and scaling in time and state, and certain types of feedback [43, 45, 76, 74].

Example [74] Consider the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{B}(\mathbf{x}, t) \ \mathbf{u}$$

and assume that there exist *control primitives* $\mathbf{u} = \mathbf{p}_i(\mathbf{x}, t)$ which, for any i, make the closed-loop system contracting in some common metric $\mathbf{M}(\mathbf{x})$. Multiplying each equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{B}(\mathbf{x}, t) \ \mathbf{p}_i(\mathbf{x}, t)$$

by a positive coefficient $\alpha_i(t)$, and summing, shows that any convex combination of the control primitives $\mathbf{p}_i(\mathbf{x}, t)$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t) + \mathbf{B}(\mathbf{x},t) \sum_i \alpha_i(t)\ \mathbf{p}_i(\mathbf{x},t) \quad , \quad \sum_i \alpha_i(t) = 1$$

also leads to a contracting dynamics in the same metric. For instance, the time-varying convex combination may correspond to smoothly blending learned primitives in a humanoid robot.

As our understanding of both brain function and robot design improves, common fundamental questions strongly suggest exploring the relations between integrative neuroscience and robotics beyond the most obvious analogies. While today the evolution and development of cognitive processes is seen as closely linked to the progressive refinement of sensorimotor functions, similarly robotics takes artificial intelligence beyond its classical conceptual domain by emphasizing the central role of physical interaction with the environment. Of course, the constraints and opportunities of robotics are very different from those of biology. While their physical hardware is far behind nature's, in principle robots can have perfect memory, near-perfect repeatibility, can use mathematics explicitly, and can simulate (imagine) specific actions much faster than humans. The travelling speed of information through an nerve axon is significantly slower than the speed of sound, while that along an electrical wire is closer to the speed of light. Processing time at each and every chemical synapse is about 1 ms, probably a major incentive for developing parallel computational architectures. But similar delay problems can also be found in robotics, if one looks not at an autonomous robot, but rather, for instance, at telerobotics over large distances.

While most of robotic theory is founded on physical models and mathematical algorithms, the fundamental conceptual tool in biology is the theory of evolution. Evolution proceeds by accumulation and combination of stable intermediate states. Conceptually, such accumulations have also been a recurrent theme in cybernetics and AI history, and also form the basis of several recent theories of brain function. However, in themselves, accumulations and combinations of stable elements have no reason to be stable. Hence our hypothesis in [76, 74, 44] that evolution will favor a contraction-like form of stability, which automatically guarantees stability in combinations, since this would considerably reduce (in effect, avoid combinatorial explosion of) trial-and-error as the systems become large and complex. Thus, contraction theory may help guide functional modelling of the central nervous system, and conversely it provides a systematic method to build arbitrarily complex robots out of simpler elements.

Incidentally, the definition of contraction fits rather naturally with known data on biological motion perturbation, e.g. perturbation of arm movement [80, 94]. Furthermore, it is intrinsic, in the sense that the system's "nominal" behavior needs not be known. Finally, such a form of stability, at least in a local sense, is also a basic prerequisite for any learning, since it guarantees the consistency of the system's behavior in the presence of small disturbances or variations in initial conditions. Automatic contraction preservation is a a property reminiscent of input-output passivity [65], although it applies in state-space to more general forms of combinations. An interesting discussion of the application of passivity tools to recursive refinement of the control of movement can be found in [2].

In [97] we use contraction to derive a nonlinear observer which computes the continuous state of an inertial navigation system based on partial discrete measurements, the so-called strap-down problem. The mathematical statement of this problem is common to aircraft navigation, robot localization, and head stabilization in mammals and birds. Indeed, the human vestibular system uses otolithic organs measuring linear acceleration and semi-circular canals estimating angular velocity through heavily damped angular acceleration signals, an information then combined with visual data at much slower update rate. Furthermore, head stabilization is likely used to simplify control and overall balance [5], a feature yet to be implemented in humanoid or flying robots.

3 Synchronization

We use partial contraction analysis to study synchronization phenomena. This section is largely based on [88] to which the reader is referred for details. Its results will be further expanded and systematized in section 4.

Theorem 2. *Consider a nonlinear system of the form*

$$\dot{\mathbf{x}} \ = \ \mathbf{f}(\mathbf{x}, \mathbf{x}, t)$$

and assume that the auxiliary system

$$\dot{\mathbf{y}} \ = \ \mathbf{f}(\mathbf{y}, \mathbf{x}, t)$$

is contracting with respect to $\mathbf{y}$*. If a particular solution of the auxiliary* $\mathbf{y}$*-system verifies a smooth specific property, then all trajectories of the original* $\mathbf{x}$*-system verify this property exponentially. The original system is said to be partially contracting.*

Proof: The virtual, observer-like $\mathbf{y}$-system has two particular solutions, namely $\mathbf{y}(t) = \mathbf{x}(t)$ for all $t \geq 0$ and the solution with the specific property. This implies that $\mathbf{x}(t)$ verifies the specific property exponentially.

Example [78] Consider a rigid robot model

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \tau \tag{2}$$

and the energy-based controller [75]

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}}_\mathbf{r} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}_\mathbf{r} + \mathbf{g}(\mathbf{q}) - \mathbf{K}(\dot{\mathbf{q}} - \dot{\mathbf{q}}_\mathbf{r}) = \tau \quad (3)$$

with $\mathbf{K}$ a constant s.p.d. matrix. The virtual $\mathbf{y}$-system

$$\mathbf{H}(\mathbf{q})\dot{\mathbf{y}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\mathbf{y} + \mathbf{g}(\mathbf{q}) - \mathbf{K}(\dot{\mathbf{q}} - \mathbf{y}) = \tau \quad (4)$$

has $\dot{\mathbf{q}}$ and $\dot{\mathbf{q}}_\mathbf{r}$ as particular solutions, and furthermore is contracting, since the skew-symmetry of the matrix $\dot{\mathbf{H}} - \mathbf{2C}$ implies

$$\frac{\mathbf{d}}{\mathbf{dt}}\ \delta\mathbf{y}^\mathbf{T}\mathbf{H}\delta\mathbf{y} = -\mathbf{2}\delta\mathbf{y}^\mathbf{T}(\mathbf{C}+\mathbf{K})\delta\mathbf{y} + \delta\mathbf{y}^\mathbf{T}\dot{\mathbf{H}}\delta\mathbf{y} = -\mathbf{2}\delta\mathbf{y}^\mathbf{T}\mathbf{K}\delta\mathbf{y}$$

Thus $\dot{\mathbf{q}}$ tends to $\dot{\mathbf{q}}_\mathbf{r}$ exponentially. Making then the usual choice $\dot{\mathbf{q}}_\mathbf{r} = \dot{\mathbf{q}}_\mathbf{d} - \lambda(\mathbf{q} - \mathbf{q}_\mathbf{d})$ creates a hierarchy and implies in turn that $\mathbf{q}$ tends to $\mathbf{q}_\mathbf{d}$ exponentially.

Example Consider a convex combination or interpolation between contracting dynamics

$$\dot{\mathbf{x}} = \sum_i \alpha_i(\mathbf{x},t)\ \mathbf{f}_i(\mathbf{x},t)$$

where the individual systems $\dot{\mathbf{x}} = \mathbf{f}_i(\mathbf{x},t)$ are contracting in a common metric $\mathbf{M}(\mathbf{x})$ and have a common trajectory $\mathbf{x}_o(t)$ (for instance an equilibrium), with all $\alpha_i(\mathbf{x},t) \geq 0$ and $\sum_i \alpha_i(\mathbf{x},t) = 1$. Then all trajectories of the system globally exponentially converge to the trajectory $\mathbf{x}_o(t)$. Indeed, the auxiliary system

$$\dot{\mathbf{y}} = \sum_i \alpha_i(\mathbf{x},t)\ \mathbf{f}_i(\mathbf{y},t)$$

is contracting (with metric $\mathbf{M}(\mathbf{y})$) and has $\mathbf{x}(t)$ and $\mathbf{x}_o(t)$ as particular solutions.

Example The main idea of the virtual system in partial contraction can be applied in a variety of ways. Consider two coupled FitzHugh-Nagumo [11, 55] neural oscillators

$$\dot{v}_1 = c(v_1 + w_1 - \tfrac{1}{3}v_1^3 + I(t)) \ + \ k\ (v_2 - v_1) \qquad \dot{w}_1 = -\tfrac{1}{c}(v_1 - a + bw_1)$$

$$\dot{v}_2 = c(v_2 + w_2 - \tfrac{1}{3}v_2^3 + I(t)) \ + \ k\ (v_1 - v_2) \qquad \dot{w}_2 = -\tfrac{1}{c}(v_2 - a + bw_2)$$

where a, b, c are strictly positive constants and $I(t)$ an external input. The above imply

$$\dot{v}_1 + (2k - c)v_1 + \tfrac{1}{3}cv_1^3 - cw_1 \ = \ \dot{v}_2 + (2k - c)v_2 + \tfrac{1}{3}cv_2^3 - cw_2$$

$$\dot{w}_1 + \tfrac{1}{c}(v_1 + bw_1) \ = \ \dot{w}_2 + \tfrac{1}{c}(v_2 + bw_2)$$

Let $g_v(t)$ be the common value of the terms in the first line, and $g_w(t)$ the common value of the terms in the second line. Both $g_v(t)$ and $g_w(t)$ depend on initial conditions. Define the virtual system

$$\dot{y}_v + (2k - c)y_v + \tfrac{1}{3}cy_v^3 - cy_w \;=\; g_v(t)$$

$$\dot{y}_w + \tfrac{1}{c}(y_v + by_w) \;=\; g_w(t)$$

By construction, both neural oscillators follow particular trajectories of the virtual system. Furthermore, using a metric transformation $\boldsymbol{\Theta} = \text{diag}(1, c)$ yields the generalized Jacobian

$$\mathbf{F} \;=\; \begin{bmatrix} -2k - c(y_v^2 - 1) & 1 \\ -1 & -\frac{b}{c} \end{bmatrix}$$

so that the virtual system is contracting if $2k > c$. Thus, if $2k > c$, the two neural oscillators synchronize with an exponential rate $\min(2k - c \;,\; b/c)$.

The results extend to global synchronization conditions for networks of locally coupled FitzHugh-Nagumo neural oscillators [88]. It can also be shown that in such networks, a *single* inhibitory link of the same gain between two arbitrary nodes can destroy synchronization. This can can provide a simple mechanism to avoid synchronization when it is undesirable. Such inhibition properties may be useful in pattern recognition to achieve rapid desynchronization between different objects. They may also be used as simplified models of minimal mechanisms for turning off unwanted synchronization, as e.g. in epileptic seizures or oscillations in internet traffic. In such applications, small and localized inhibition may also allow one to destroy unwanted synchronization while only introducing a small disturbance to the nominal behavior of the system.

Cascades of inhibition are common in the brain, in a way perhaps reminiscent of NAND-based logic.

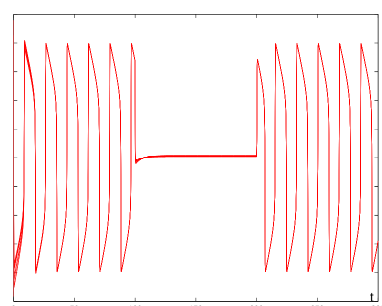

Fig. 1. An example of fast inhibition with a single inhibitory link. The plot shows the states of ten FitzHugh-Nagumo neurons as functions of time. The inhibitory link is actived at $t = 100$ and removed at $t = 200$. Thus, synchronization can be readily achieved and just as readily turned off.

Distributed synchronization phenomena are the subject of intense research. In the brain such phenomena are known to occur at different scales, and are heavily studied at both the anatomical and computational levels. In particular, synchronization has been proposed as a general principle for temporal binding of multisensory data [73, 20, 41, 53, 84, 37, 57], and as a mechanism for perceptual grouping [95], neural computation [6, 7, 92] and neural communication [34, 27, 69, 70]. The recently recognized pervasiveness [13] of diffusion-like, bilateral electrical synapses (gap junctions) makes these synchronization models particularly intriguing. Not entirely coincidentally, the same basic mathematics describing the collective behavior of neurons also describe the collective behavior of fish schools or bird flocks, as well as certain types of phase-transition in physics [82].

4 Polyrhythms

This section, which extends and systematizes the results of the section 3, is based on [63], to which the reader is referred for details.

4.1 Concurrent Synchronization

In an ensemble of dynamical elements, concurrent synchronization is defined as a regime where the whole system is divided into multiple groups of fully synchronized elements[1], but elements from different groups are not necessarily synchronized [4, 96, 64] and can be of entirely different dynamics [16]. It can be easily shown that such a regime corresponds to a flow-invariant linear subspace of the global state space. Concurrent synchronization phenomena are likely pervasive in the brain, where multiple "rhythms" are known to coexist [34, 69], neurons can exhibit many qualitatively different types of oscillations [34, 26], and functional models often combine multiple oscillatory dynamics.

A simple sufficient condition for a general dynamical system to converge to a flow-invariant subspace is introduced in [63]. The analysis is built upon nonlinear contraction theory [43, 88], and thus it inherits many of the theory's features :

- global exponential convergence and stability are guaranteed, as opposed to [4, 96, 17] where only stability in the neighborhood of the invariant manifold is discussed,
- convergence rates can be explicitly computed as eigenvalues of well-defined symmetric matrices,
- under simple conditions, convergence to a concurrently synchronized state can be preserved through system combinations,

[1] In the literature, this phenomenon is often called *poly-* or *partial* synchronization. However, the latter term can also designate a regime where the elements are not fully synchronized but behave coherently [82].

- generalized input-symmetries in the sense of [16] can be systematically exploited,
- robustness to variations in dynamics can be easily quantified.

Consider, in $\mathbb{R}^n$, the deterministic system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{5}$$

where $\mathbf{f}$ is a smooth nonlinear function. Assume that there exists a *flow-invariant linear subspace* $\mathcal{M}$ (i.e. a linear subspace $\mathcal{M}$ such that $\forall t : \mathbf{f}(\mathcal{M}, t) \subset \mathcal{M}$), which implies that any trajectory starting in $\mathcal{M}$ remains in $\mathcal{M}$. Let $p = \dim(\mathcal{M})$, and consider an orthonormal basis $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ where the first p vectors form a basis of $\mathcal{M}$ and the last $n - p$ a basis of $\mathcal{M}^\perp$. Define an $(n-p) \times n$ matrix $\mathbf{V}$ whose rows are $\mathbf{e}_{p+1}^\top, \ldots, \mathbf{e}_n^\top$. $\mathbf{V}$ may be regarded as a projection[2] on $\mathcal{M}^\perp$, and it verifies [24, 31] :

$$\mathbf{V}^\top\mathbf{V} + \mathbf{U}^\top\mathbf{U} = \mathbf{I}_n \qquad \mathbf{V}\mathbf{V}^\top = \mathbf{I}_{n-p} \qquad \mathbf{x} \in \mathcal{M} \iff \mathbf{V}\mathbf{x} = \mathbf{0}$$

where $\mathbf{U}$ is the matrix formed by the first p vectors.

Now let $\mathbf{z} = \mathbf{V}\mathbf{x}$. By construction, $\mathbf{x}$ converges to the subspace $\mathcal{M}$ if and only if $\mathbf{z}$ converges to $\mathbf{0}$. Multiplying (5) by $\mathbf{V}$ on the left, we get

$$\dot{\mathbf{z}} = \mathbf{V}\mathbf{f}(\mathbf{V}^\top\mathbf{z} + \mathbf{U}^\top\mathbf{U}\mathbf{x}, t)$$

Construct the auxiliary system

$$\dot{\mathbf{y}} = \mathbf{V}\mathbf{f}(\mathbf{V}^\top\mathbf{y} + \mathbf{U}^\top\mathbf{U}\mathbf{x}, t) \tag{6}$$

By construction, a particular solution of system (6) is $\mathbf{y}(t) = \mathbf{z}(t)$. In addition, since $\mathbf{U}^\top\mathbf{U}\mathbf{x} \in \mathcal{M}$ and $\mathcal{M}$ is flow-invariant, $\mathbf{f}(\mathbf{U}^\top\mathbf{U}\mathbf{x}) \in \mathcal{M} = \mathrm{Null}(\mathbf{V})$. Thus $\mathbf{y}(t) = \mathbf{0}$ is another particular solution of system (6). If furthermore system (6) is contracting with respect to $\mathbf{y}$, then $\mathbf{z}(t)$ will converge exponentially to $\mathbf{0}$. This leads to [63]

Theorem 3. *If a linear subspace $\mathcal{M}$ is flow-invariant and if system (6) is contracting, then all solutions of system (5) converge exponentially to $\mathcal{M}$.*

In practice, the subspace $\mathcal{M}$ is often defined by the conjunction of $(n-p)$ linear constraints. In a synchronization context, each of the constraints may be, e.g., of the form $\mathbf{x}_i = \mathbf{x}_j$ where $\mathbf{x}_i$ and $\mathbf{x}_j$ are subvectors of the state $\mathbf{x}$. This provides directly a (generally not orthonormal) basis $(\mathbf{e}'_{p+1}, \ldots, \mathbf{e}'_n)$ of $\mathcal{M}^\perp$, and thus a matrix $\mathbf{V}'$ whose rows are ${\mathbf{e}'_{p+1}}^\top, \ldots, {\mathbf{e}'_n}^\top$, and which verifies

$$\mathbf{V}' = \mathbf{T}\mathbf{V}$$

[2] For simplicity we shall call $\mathbf{V}$ a “projection”, although the actual projection matrix is in fact $\mathbf{V}^\top\mathbf{V}$.

with $\mathbf{T}$ an invertible $(n-p) \times (n-p)$ matrix, and $\mathbf{x} \in \mathcal{M} \iff \mathbf{V}'\mathbf{x} = \mathbf{0}$. For instance, for three systems, each of dimension m and state $\mathbf{x}_i$, one has

$$\mathbf{V}' = \begin{pmatrix} \mathbf{I}_m & -\mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m & -\mathbf{I}_m \end{pmatrix}$$

Thus, using an identity metric in (6), the sufficient condition for convergence to $\mathcal{M}$

$$\forall\ \mathbf{x}, \qquad \mathbf{V} \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right) \mathbf{V}^{\top} < \mathbf{0} \tag{7}$$

can be written equivalently in the more simply evaluated form

$$\forall\ \mathbf{x}, \qquad \mathbf{V}' \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right) \mathbf{V}'^{\top} < \mathbf{0} \tag{8}$$

Note however that to evaluate explicitly the convergence rate to $\mathcal{M}$, one has to use the orthonormal version, e.g. through a Gram-Schmidt procedure [24] on the rows of $\mathbf{V}'$.

Different "rhythms" $(\alpha, \beta, \gamma, \delta)$ are known to coexist in the brain, which, in the light of the previous analysis, may be interpreted and modelled as concurrently synchronized regimes. Since contracting systems driven by periodic inputs will have states of the same period [43], different but synchronized computations could be robustly carried out by specialized areas in the brain using synchronized elements as their inputs. Such a temporal "binding" [73, 20, 41, 84, 37, 57, 95, 8, 13] mechanism would also complement the general argument in [76] that multisensory integration may occur through the interaction of contracting computational systems connected through an extensive network of feedback loops. Furthermore, because of the preservation of contraction through system combinations, Theorem 3 suggests a mechanism for stable accumulation and interaction of concurrently synchronized groups, showing [63] that the simple conditions for contraction to a linear subspace, combined with the high fan-out of typical neurons, increase the plausibility of large concurrently synchronized structures being created in the central nervous system in the course of evolution and development. Making these observations precise is the subject of current research.

4.2 Percolation

When a sync subspace is a strict superset of another, one should expect that the convergence to the former state is "easier" than the convergence to the latter [96, 4, 64]. This progressive synchronization or "percolation" effect can be quantified easily from Theorem 3, by noticing that

$$\mathcal{M}_A \supset \mathcal{M}_B \quad \Rightarrow \quad \mathcal{M}_A^{\perp} \subset \mathcal{M}_B^{\perp} \quad \Rightarrow \quad \lambda_{\min}(\mathbf{V}_A \mathbf{L}_s \mathbf{V}_A^{\top}) \geq \lambda_{\min}(\mathbf{V}_B \mathbf{L}_s \mathbf{V}_B^{\top})$$

While in the case of identical systems and relatively uniform topologies, this effect may often be too fast to observe, the above applies to the general concurrent synchronisation case and quantifies the associated and possibly very distinct time-scales.

4.3 Coincidence and Symmetry Detection

Coincidence detection is a classic mechanism proposed for segmentation and classification. In an image for instance, elements moving at a common velocity are typically interpreted as being part of a single object, and this even when the image is only composed of random dots [41, 37].

The possibility of decentralized synchronization via central diffusive couplings can be used in building a coincidence detector. In [92], inspired in part by [7], the authors consider a leader-followers network of FitzHugh-Nagumo oscillators, where each follower oscillator i receives an external input I_i as well as a diffusive coupling from the leader oscillator (the element e_1 of G_1). Oscillators i and j receiving the same input ($I_i = I_j$) synchronize, so that choosing the system output as $\sum_{1 \leq i \leq n} [\dot{v}_i]^+$ captures the moment when a large number of oscillators receive the same input.

However, the previous development also implies that this very network can detect the moments when *several* groups of identical inputs exist. Furthermore, it is possible to identify the number of such groups and their relative size. Indeed, assume that the inputs are divided into k groups, such that for each group G_m, one has $\forall i, j \in G_m, I_i = I_j$. Since the oscillators in G_m only receive as input (a) the output of the leader, which is the same for everybody and (b) the external input I_i, which is the same for every oscillator in group G_m, they are input-symmetric and should synchronize with each other.

Symmetry, in particular bilateral symmetry, has also been shown to play a key role in human perception [6]. Consider a group of oscillators having the same individual dynamics and connected together in a symmetric manner. If we present to the network an input having the same symmetry, some of the oscillators will synchronize as predicted by the earlier theoretical results. One application of this idea is to build fast symmetry detectors, extending the oscillator-based coincidence detectors of the previous section [63].

4.4 Robustness

Robustness results for contracting systems [43] can be exploited to guarantee approximate synchronization even when the elements are not exactly identical, with important practical implications for the synchronization of spiking neurons of different dynamics.

Consider again a network of n dynamical elements

$$\dot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}_i, t) + \sum_{j \neq i} \mathbf{K}_{ij}(\mathbf{x}_j - \mathbf{x}_i) \qquad i = 1, \ldots, n \tag{9}$$

with now possibly $\mathbf{f}_i \neq \mathbf{f}_j$ for $i \neq j$, and let us apply the robustness result for contracting systems of [43] to the projected system of Theorem 3. For strong enough coupling strength, all trajectories of the system will exponentially converge to a boundary layer of thickness D/λ around the sync subspace $\mathcal{M}$, where λ is the contraction rate of the auxiliary system and D is a measure of the dissimilarity of the elements [63].

Consider for instance, a system of the form (similar to the model used for coincidence detection in [88] and the previous section)

$$\dot{x}_i = f(x_i) + I_i + k(x_0 - x_i)$$

In this case, $D = \frac{I_{\max} - I_{\min}}{2}$.

Assume now that two spiking neurons are approximately synchronized, as just discussed. Then, since spiking induces large abrupt variations, the neurons must spike approximately at the same time. More specifically, if the bound on their trajectory discrepancy guaranteed by the above robustness result is significantly smaller than spike size, then this bound will automatically imply that the two neurons spike approximately at the same time.

4.5 Locomotion

In an animal/robotics locomotion context, central pattern generators are often modelled as coupled nonlinear oscillators delivering phase-locked signals. Consider for instance [63] a system of three coupled 2-dimensional Andronov-Hopf oscillators,

$$\begin{cases} \dot{\mathbf{x}}_1 = \mathbf{f}(\mathbf{x}_1) + k(\mathbf{R}_{\frac{2\pi}{3}}\mathbf{x}_2 - \mathbf{x}_1) \\ \dot{\mathbf{x}}_2 = \mathbf{f}(\mathbf{x}_2) + k(\mathbf{R}_{\frac{2\pi}{3}}\mathbf{x}_3 - \mathbf{x}_2) \\ \dot{\mathbf{x}}_3 = \mathbf{f}(\mathbf{x}_3) + k(\mathbf{R}_{\frac{2\pi}{3}}\mathbf{x}_1 - \mathbf{x}_3) \end{cases}$$

where $\mathbf{f}$ is the dynamics of an Andronov-Hopf oscillator and the matrix $\mathbf{R}_{\frac{2\pi}{3}}$ describes a $\frac{2\pi}{3}$ planar rotation :

$$\mathbf{f}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u - v - u^3 - uv^2 \\ u + v - v^3 - vu^2 \end{pmatrix} \qquad \mathbf{R}_{\frac{2\pi}{3}} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

We can rewrite the dynamics as $\dot{\mathbf{x}}_{\{\}} = \mathbf{f}_{\{\}}(\mathbf{x}_{\{\}}) - k\mathbf{L}\mathbf{x}_{\{\}}$, where

$$\mathbf{L} = \begin{pmatrix} \mathbf{I}_2 & -\mathbf{R}_{\frac{2\pi}{3}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 & -\mathbf{R}_{\frac{2\pi}{3}} \\ -\mathbf{R}_{\frac{2\pi}{3}} & \mathbf{0} & \mathbf{I}_2 \end{pmatrix}$$

First, observe that the *linear* subspace $\mathcal{M} = \left\{ \left(\mathbf{R}^2_{\frac{2\pi}{3}}(\mathbf{x}), \mathbf{R}_{\frac{2\pi}{3}}(\mathbf{x}), \mathbf{x} \right) : \mathbf{x} \in \mathbb{R}^2 \right\}$ is flow-invariant, and that $\mathcal{M}$ is also a subset of Null($\mathbf{L}_s$). Next, remark that the characteristic polynomial of $\mathbf{L}_s$ is $X^2\,(X - 3/2)^4$ so that the eigenvalues of $\mathbf{L}_s$ are 0, with multiplicity 2, and 3/2, with multiplicity 4. Now since $\mathcal{M}$ is 2-dimensional, it is exactly the nullspace of $\mathbf{L}_s$, which implies in turn that $\mathcal{M}^\perp$ is the eigenspace corresponding to the eigenvalue 3/2.

Moreover, the eigenvalues of $\mathbf{J}_s(u,v)$ are $1 - (u^2 + v^2)$ and $1 - 3(u^2 + v^2)$, which are upper-bounded by 1. Thus, using the previous development, for

$k > 2/3$ the three systems will *globally exponentially converge to a* $\frac{2\pi}{3}$*-phase-locked state* (i. e. a state in which the difference of the phases of two consecutive elements is constant and equals $\frac{2\pi}{3}$).

4.6 Coupled CPGs

Let us further illustrate combinations of such systems on the following example, based on [71], which studies models of central pattern generators in fish and salamanders [25].

Consider again an Andronov-Hopf oscillator, now with a limit cycle of constant radius $\rho > 0$,

$$\dot{\mathbf{x}} = \mathbf{f}_\rho(\mathbf{x}) = \begin{pmatrix} -v - \left(\frac{u^2+v^2}{\rho^2} - 1\right) u \\ u - \left(\frac{u^2+v^2}{\rho^2} - 1\right) v \end{pmatrix},$$

where $\mathbf{x}^{\mathrm{T}} = [u,\ v]$. Note that

- $\mathbf{f}_\rho(\mathbf{R}\mathbf{x}) = \mathbf{R}\mathbf{f}_\rho(\mathbf{x})$ for an arbitrary rotation $\mathbf{R}$
- $\mathbf{f}_\rho(k\mathbf{x}) = k\mathbf{f}_{\rho/k}(\mathbf{x})$ for $k > 0$,

Consider now a two-way chain of n such oscillators with *phase-shift diffusive couplings*,

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{f}_1(\mathbf{x}_1) - \gamma b_1(\mathbf{x}_1 - \mathbf{T}_1^{-1}\mathbf{x}_2) \\ &\ \vdots \\ \dot{\mathbf{x}}_i &= \mathbf{f}_i(\mathbf{x}_i) - \gamma a_{i-1}(\mathbf{x}_i - \mathbf{T}_{i-1}\mathbf{x}_{i-1}) - \gamma b_i(\mathbf{x}_i - \mathbf{T}_i^{-1}\mathbf{x}_{i+1}) \\ &\ \vdots \\ \dot{\mathbf{x}}_n &= \mathbf{f}_n(\mathbf{x}_n) - \gamma a_{n-1}(\mathbf{x}_n - \mathbf{T}_{n-1}\mathbf{x}_{n-1}) \end{aligned} \tag{10}$$

where each $\rho_i > 0$ is the radius of the corresponding limit cycle (the $\rho_i > 0$ may be distinct) and $\mathbf{T}_i$ is defined as

$$\mathbf{T}_i = \frac{\rho_{i+1}}{\rho_i}\mathbf{R}_i,$$

with $\mathbf{R}_i$ a planar rotation of ϕ_i,

$$\mathbf{R}_i = \begin{pmatrix} \cos(\phi_i) & -\sin(\phi_i) \\ \sin(\phi_i) & \cos(\phi_i) \end{pmatrix}.$$

Hence, a pair of diffusive couplings $(\mathbf{x}_{i+1} - \mathbf{T}_i\mathbf{x}_i)$ and $(\mathbf{x}_i - \mathbf{T}_i^{-1}\mathbf{x}_{i+1})$ matches different limit cycle radii ρ_{i+1} and ρ_i, and shifts the phase as much as ϕ_i. The coupling is two-way in that the *downward* coupling $(\mathbf{x}_{i+1} - \mathbf{T}_i\mathbf{x}_i)$ that appears in the equation of $\mathbf{x}_{i+1}$ oscillator pushes $\mathbf{x}_{i+1}$ to follow $\mathbf{x}_i$ adjusted through $\mathbf{T}_i$; and the *upward* coupling $(\mathbf{x}_i - \mathbf{T}_i^{-1}\mathbf{x}_{i+1})$ pushes $\mathbf{x}_i$ to follow $\mathbf{x}_{i+1}$ through $\mathbf{T}_i^{-1}$. γa_i is the downward coupling strength and γb_i the upward. By combining the coupling coefficients multiplied by γ, we can adjust the overall strength without affecting the upward to downward ratio.

Defining collective quantities

$$\mathbf{x}_{\{\}} \stackrel{\Delta}{=} \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{pmatrix} \qquad f_{\{\}}(\mathbf{x}_{\{\}}) = \begin{pmatrix} \mathbf{f}_1(\mathbf{x}_1) \\ \vdots \\ \mathbf{f}_n(\mathbf{x}_n) \end{pmatrix}$$

we can rewrite the system (10) as

$$\dot{\mathbf{x}}_{\{\}} \stackrel{\Delta}{=} \mathbf{f}_{\{\}}(\mathbf{x}_{\{\}}) - \gamma \mathbf{L} \mathbf{x}_{\{\}}, \tag{11}$$

where the coupling matrix $\mathbf{L}$ is then defined as

$$\mathbf{L} = \begin{pmatrix} b_1\mathbf{I} & -b_1\mathbf{T}_1^{-1} & \mathbf{0} \\ -a_1\mathbf{T}_1 & a_1\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} + \ldots + \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & b_{n-1}\mathbf{I} & -b_{n-1}\mathbf{T}_{n-1}^{-1} \\ \mathbf{0} & -a_{n-1}\mathbf{T}_{n-1} & a_{n-1}\mathbf{I} \end{pmatrix}.$$

In order to make the coupling matrix $\mathbf{L}$ symmetric, we define a coordinate transformation $\mathbf{y}_{\{\}} = \mathbf{\Phi}\mathbf{x}_{\{\}}$ where

$$\mathbf{\Phi} = \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \frac{\rho_1}{\rho_2}\sqrt{\frac{b_1}{a_1}}\mathbf{I} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \frac{\rho_1}{\rho_3}\sqrt{\frac{b_1 b_2}{a_1 a_2}}\mathbf{I} & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Then (11) becomes

$$\dot{\mathbf{y}}_{\{\}} = \mathbf{\Phi}\mathbf{f}_{\{\}}(\mathbf{\Phi}^{-1}\mathbf{y}_{\{\}}) - \gamma\mathbf{\Phi}\mathbf{L}\mathbf{\Phi}^{-1}\mathbf{y}_{\{\}}$$

where the Jacobian of the couplings is symmetric positive semidefinite,

$$\mathbf{\Phi}\mathbf{L}\mathbf{\Phi}^{-1} = \begin{pmatrix} b_1\mathbf{I} & -\sqrt{a_1 b_1}\mathbf{R}_1^{\mathrm{T}} & \mathbf{0} \cdots \\ -\sqrt{a_1 b_1}\mathbf{R}_1 & a_1\mathbf{I} & \mathbf{0} \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \cdots \\ \vdots & \vdots & \vdots \ddots \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \cdots \\ \mathbf{0} & b_2\mathbf{I} & -\sqrt{a_2 b_2}\mathbf{R}_2^{\mathrm{T}} & \mathbf{0} \cdots \\ \mathbf{0} & -\sqrt{a_2 b_2}\mathbf{R}_2 & a_2\mathbf{I} & \mathbf{0} \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \cdots \\ \vdots & \vdots & \vdots & \vdots \ddots \end{pmatrix} + \cdots$$

The corresponding linear invariant subspace in $\mathbf{y}$-system is

$$\mathcal{M}_y = \{\sqrt{b_i}\mathbf{R}_i\mathbf{y}_i - \sqrt{a_i}\mathbf{y}_{i+1} = \mathbf{0},\ i = 1, \ldots\ n-1\},$$

or equivalently, in the original $\mathbf{x}$-system,

$$\mathcal{M}_x = \{\mathbf{x}_{i+1} = \mathbf{T}_i\mathbf{x}_i\ ,\ i = 1, \ldots\ n-1\}$$

Notice that $\mathcal{M}_y$ is the eigenspace corresponding to zero eigenvalues of $\mathbf{\Phi L \Phi}^{-1}$.

We can now construct a non-orthonormal basis for $\mathcal{M}_y^{\perp}$ directly from $\mathcal{M}_y$ as

$$\widetilde{\mathbf{V}} = \begin{pmatrix} \sqrt{b_1}\mathbf{R}_1 & -\sqrt{a_1}\mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \sqrt{b_2}\mathbf{R}_2 & -\sqrt{a_2}\mathbf{I} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \sqrt{b_{n-1}}\mathbf{R}_{n-1} & -\sqrt{a_{n-1}}\mathbf{I} \end{pmatrix},$$

whose rows form a basis of $\mathcal{M}_y^{\perp}$. Through orthogonalization of $\widetilde{\mathbf{V}}$ we obtain the orthogonal matrix $\mathbf{V}$, and the resulting $\mathbf{V}\Phi\mathbf{L}\Phi^{-1}\mathbf{V}^{\mathrm{T}}$ is positive definite. The Jacobian of $\dot{\mathbf{x}} = \mathbf{f}_\rho(\mathbf{x})$ is

$$\frac{\partial \mathbf{f}_\rho}{\partial \mathbf{x}} = \begin{pmatrix} -2\frac{u^2}{\rho^2} - \frac{u^2+v^2}{\rho^2} + 1 & -1 - 2\frac{uv}{\rho^2} \\ 1 - 2\frac{uv}{\rho^2} & -2\frac{v^2}{\rho^2} - \frac{u^2+v^2}{\rho^2} + 1 \end{pmatrix}$$

and the eigenvalues of its symmetric part are

$$1 - \frac{u^2+v^2}{\rho^2};\ 1 - \frac{3(u^2+v^2)}{\rho^2}$$

Hence, by choosing γ large enough so that

$$\gamma \ \ \lambda_{\min}(\mathbf{V}\Phi\mathbf{L}\Phi^{-1}\mathbf{V}^{\mathrm{T}}) \ > \ 1$$

the generalized projected Jacobian verifies

$$\mathbf{V}\left[\Phi\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} - \gamma\mathbf{L}\right)\Phi^{-1}\right]_s \mathbf{V}^{\mathrm{T}} < 0.$$

Thus, a two-way chain of oscillators with arbitrary positive couplings gains can be synchronized to the desired invariant set $\mathcal{M}_x$ for sufficiently large overall coupling strength γ. The analysis extends easily [71] to double antisynchronized chains and to the other types of gaits studied e.g. in [25].

4.7 Frequency-Based Gait Selection

To study concurrent synchronization, extensive use is made in [63] of generalized symmetries [16]. Actually, replacing ordinary connections in the CPGs by filters enables *frequency-based* symmetry selection. This idea may have powerful applications, one of which could be automatic gait selection in locomotion. An simplified analogy with horse gaits can be made, for instance, by associating the low-frequency regime with the walk (left fore, right hind, right fore, left hind), and the high-frequency regime with the trot (left fore and right hind simultaneously, then right fore and left hind simultaneously). Transitions between the two regimes can occur automatically according to the speed of the horse (the frequency of its gait).

Note that standard techniques allow sharp causal filters with frequency-independent delays to be easily constructed [61].

References

1. Amari S, Arbib M (1977) Competition and Cooperation in Neural Nets. In: Metzler J (ed) Systems Neuroscience. Academic Press, San Diego, 119–165
2. Arimoto S, Naniwa T (2002) Learnability and Adaptability from the Viewpoint of Passivity Analysis, *Intelligent Automation and Soft Computing* 8(2)
3. Ashby WR (1966) In: Design for a Brain: the Origin of Adaptive Behavior. 2nd edition, Science Paperbacks
4. Belykh V, Belykh I, Mosekilde E (2001) Cluster Synchronization Modes in an Ensemble of Coupled Chaotic Oscillators. *Phys. Rev. E*
5. Berthoz A (1999) The Brain's Sense of Movement. Harvard University Press, Cambridge Massachusetts
6. Braitenberg V (1984) *Vehicles: Experiments in Synthetic Psychology*, chap. 9. The MIT Press
7. Brody C, Hopfield J (2003) Simple Networks for Spike-Timing-Based Computation with Application to Olfactor Processing. *Neuron*
8. Crick F, Koch C (2005) What is the Function of the Claustrum? *Phil. Trans. Roy. Soc. Lond. B, 360*
9. Dayan P, Hinton G, Neal R, Zemel R (1995) The Helmholtz Machine, *Neural Computation, 7*
10. Fiedler M(1976) Algebraic Connectivity of Graphs, *Czechoslovak Mathematical Journal*
11. FitzHugh R (1961) Impulses and Physiological States in Theoretical Models of Nerve Membrane. Biophys. Journal 1:445–466
12. Fujisawa S, Matsuki N, Ikegaya Y (2006) Single Neurons Can Induce Phase Transitions of Cortical Recurrent Networks with Multiple Internal St ates. *Cerebral Cortex* 16
13. Fukuda T, Kosaka T, Singer W, Galuske RAW (2006) Gap Junctions among Dendrites of Cortical GABAergic Neurons Establish a Dense and Widespread Intercolumnar Network. *The Journal of Neuroscience* 26(13)
14. George D, Hawkins, J (2005) Invariant Pattern Recognition using Bayesian Inference on Hierarchical Sequences
15. Gerstner W (2001) A Framework for Spiking Neuron Models: The Spike Response Model. In: The Handbook of Biological Physics. 4:469–516
16. Golubitsky M, Stewart I (2003) Synchrony versus Symmetry in Coupled Cells. *Equadiff 2003: Proceedings of the International Conference on Differential Equations*
17. Golubitsky M, Stewart I, Török A (2005) Patterns of Symmetry in Coupled Cell Networks with Multiple Arrows. *SIAM J. Appl. Dynam. Sys.*
18. Gray CM (1999) The Temporal Correlation Hypothesis of Visual Feature Integration: Still Alive and Well. Neuron 24:31–47
19. Grossberg S (1982) *Studies of Mind and Brain*, Kluwer
20. Grossberg S (2000) The Complementary Brain: A Unifying View of Brain Specialization and Modularity. *Trends in Cognitive Sciences*
21. Guckenheimer J, Holmes P (1983) Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, *Springer Verlag*
22. Hodgkin AL, Huxley AF (1952) A Quantitative Description of Membrane Current and its Application to Conduction and Excitation in Nerve, *J. Physiol.* 117:500

23. Hopfield JJ, Brody CD (2001) What Is A Moment? Transient Synchrony as a Collective Mechanism for Spatiotemporal Integration. Proc. Natl. Acad. Sci. USA 98:1282–1287
24. Horn R, Johnson C (1985) *Matrix Analysis*, Cambridge University Press
25. Ijspeert A, Crespi A, Cabelguen J-M (2005) Simulation and Robotic Studies of Salamander Locomotion: Applying Neurobiological Principles to the Control of Locomotion in Robots
26. Izhikevich E (2003) Simple Model of Spiking Neuron, *IEEE Trans. on Neural Networks*
27. Izhikevich E, Desai N, Walcott E, Hoppensteadt F (2003) Bursts as a Unit of Neural Information: Selective Communication via Resonance, *Trends in Neuroscience* 26(3)
28. Izhikevich E (2004) Which Model to Use for Cortical Spiking Neurons? *IEEE Transactions on Neural Networks*
29. Izhikevich E, Kuramoto Y (2006) Weakly Coupled Oscillators. *Encyclopedia of Mathematical Physics*, Elsevier
30. Jadbabaie A, Lin J, Morse A (2003) Coordination of Groups of Mobile Autonomous Agents using Nearest Neighbor Rules. IEEE Transactions on Automatic Control 48:988-1001
31. Jadbabaie A, Motee N, Barahona M (2004) On the Stability of the Kuramoto Model of Coupled Nonlinear Oscillators. *Proceedings of the American Control Conf.*
32. Jin DZ, Seung HS (2002) Fast Computation With Spikes in a Recurrent Neural Network. Physical Review E 65:051922
33. Jouffroy J, Slotine J-JE (2004) Methodological Remarks on Contraction Theory, *IEEE CDC*
34. Kandel E, Schwartz J, Jessel T (2006) *Principles of Neural Science*, 5th ed., McGraw-Hill
35. Khalil H (1995) Nonlinear Systems, 2nd Ed.,*Prentice-Hall*
36. Knuth D (1997) *The Art of Computer Programming*, 3rd Ed. Addison-Wesley
37. Koch C (2004) *The Quest for Consciousness*, Roberts and Company Publishers
38. Korner E, Gewaltig M-O, Korner U, Richter A, Rodemann T. (1999) A model of computation in neocortical architecture, *Neural Networks* 12(7-8)
39. Kumar V, Leonard N, Morse S, (eds) (2005) Cooperative Control, *Springer Lecture Notes in Control and Information Science* 309
40. Leonard NE, Fiorelli E (2001) Virtual Leaders, Artificial Potentials and Coordinated Control of Groups, *40th IEEE Conference on Decision and Control*
41. Llinas R, Leznik E, Urbano F (2002) Temporal Binding via Cortical Coincidence Detection. *PNAS 99,1*
42. Lin Z, Broucke M, Francis B (2004) Local Control Strategies for Groups of Mobile Autonomous Agents. *IEEE Trans. on Automatic Control*
43. Lohmiller W, Slotine J-JE (1998) On Contraction Analysis for Nonlinear Systems. *Automatica* 34(6)
44. Lohmiller W, Slotine J-JE (2000) Control System Design for Mechanical Systems Using Contraction Theory, *IEEE Trans. Aut. Control* 45(5)
45. Lohmiller W, Slotine J-JE (2000) Nonlinear Process Control Using Contraction Theory. *A.I.Ch.E. Journal*
46. Luettgen M, Willsky A (1995) Likelihood Calculation for a Class of Multiscale Stochastic Models, with Application to Texture Discrimination. *IEEE Transactions on Image Processing* 4(2)

47. Maass W (2000) On the Computational Power of Winner-Take-All. Neural Comput. 12(11):2519–2535
48. von der Malsburg C (1995) Binding in Models of Perception and Brain Function. Current Opinion in Neurobiology 5:520–526
49. Mataric M (1998) Coordination and Learning in Multi-Robot Systems, *IEEE Intelligent Systems* 6-8
50. May RM, Gupta S, McLean AR (2001) Infectious Disease Dynamics: What Characterizes a Successful Invader? *Phil. Trans. R. Soc. Lond. B* 356:901–910
51. Milo R, Shen-Orr S, Itzkovitz S, Kashtan N, Chklovskii D, Alon U (2002) Network Motifs: Simple Building Blocks of Complex Networks. *Science* 298
52. Mohar B (1991) Eigenvalues, Diameter, and Mean Distance in Graphs. Graphs and Combinatorics 7:53–64
53. Mountcastle V (1998) The Cerebral Cortex, *Harvard University Press*
54. Murray JD (1993) Mathematical Biology, *Berlin;New York: Springer-Verlag*
55. Nagumo J, Arimoto S, Yoshizawa S (1962) An Active Pulse Transmission Line Simulating Nerve Axon. *Proc. Inst. Radio Engineers*
56. Nayfeh AH, Blachandran B (1995) Applied Nonlinear Dynamics, *Wiley*
57. Niessing J, Ebisch B, Schmidt KE, Niessing M, Singer W, Galuske RA (2005) Hemodynamic Signals Correlate Tightly with Synchronized Gamma Oscillations. *Science, 309*
58. Nijmeijer H (2001) A dynamical control view on synchronisation. Physica D 154:219–228
59. Nowak MA, Sigmund K (2004) Evolutionary Dynamics of Biological Games. Science 303:793–799
60. Olfati-Saber R, Murray R (2004) Consensus Problems in Networks of Agents With Switching Topology and Time-Delays. *IEEE Transactions on Automatic Control*
61. Oppenheim A, Schafer R, Buck J (1999) *Discrete-Time Signal Processing, 2nd Edition* , Prentice-Hall
62. Pecora LM, Carroll TL, Johnson GA, Mar DJ (1997) Fundamentals of synchronisation in chaotic systems. Chaos 7(4)
63. Pham Q-C, Slotine J-JE (2005) Stable Concurrent Synchronization in Dynamic System Networks, *http://arxiv.org/abs/q-bio.NC/0510051*
64. Pogromsky A, Santoboni G, Nijmeijer H (2002) Partial Synchronization: from Symmetry towards Stability. *Physica D*
65. Popov VM (1973) Hyperstability of Control Systems, *Springer-Verlag*
66. Rao R, Ballard D (1999) Predictive Coding in the Visual Cortex. *Nature Neuroscience* 2(1)
67. Rao R (2005) Bayesian inference and attention in the visual cortex, *Neuroreport* 16(16)
68. Schaal S (1999) Is imitation learning the route to humanoid robots? *Trends in Cognitive Sciences* 3(6)
69. Schnitzler A, Gross J (2005) Normal and Pathological Oscillatory Communication in the Brain. *Nat. Rev. Neurosci.*
70. Schoffelen J-M, Oostenveld R, Fries P (2005) Neuronal Coherence as a Mechanism of Effective Corticospinal Interaction. *Science*
71. Seo K, Slotine J-JE (2006) Models of Global Synchronization in Fish and Salamanders, *MIT NSL Report 050106*

72. Sepulchre R, Paley D, Leonard N (2003) Collective Motion and Oscillator Synchronization. *Proceedings of the 2003 Block Island Workshop on Cooperative Control.*
73. Singer W, Gray CM (1995) Visual Feature Integration and The Temporal Correlation Hypothesis, Annu. Rev. Neurosci. 18:555-586
74. Slotine J-JE (2003) Modular Stability Tools for Distributed Computation and Control. *Int. J. Adaptive Control and Signal Processing*, 17(6)
75. Slotine J-JE, Li W (1991) Applied Nonlinear Control, *Prentice-Hall*
76. Slotine J-JE, Lohmiller W (2001) Modularity, Evolution,and the Binding Problem: A View from Stability Theory. *Neural Networks* 14
77. Slotine J-JE, Wang W (2003) A Study of Synchronization and Group Cooperation Using Partial Contraction Theory, *Block Island Workshop on Cooperative Control*, Morse S., Editor, Springer-Verlag
78. Slotine J-JE, Wang W, Elrifai K (2004) Contraction Analysis of Synchronisation and Desynchronisation in Networks of Nonlinearly Coupled Oscillators. *Proceedings of MTNS 2004*
79. Smale S (1976) A Mathematical Model of Two Cells via Turing's Equation. In: The Hopf Bifurcation and Its Applications. Spinger-Verlag 354–367
80. Soechting J, Lacquaniti F (1988) Quantitative evaluation of the electromyographic responses to multidirectional load perturbations of the human arm. *J Neurophysiol.* 59(4)
81. Strogatz SH (1994) Nonlinear Dynamics and Chaos: with applications to physics, biology, chemistry, and engineering. Addison-Wesley Pub., Reading, MA
82. Strogatz SH (2000) From Kuramoto to Crawford: Exploring the Onset of Synchronization in Populations of Coupled Oscillators. *Physica D*
83. Thorpe S, Delorme A, Rullen RV (2001) Spike-Based Strategies for Rapid Processing. Neural Networks 14:715–725
84. Tononi G, et al. (1998) Proc. Natl. Acad. Sci. USA 95:3198–3203
85. Turing A (1952) The Chemical Basis of Morphogenesis. Philos. Trans. Roy. Soc. B 237:37–72
86. Walter WG (1950) An Imitation of Life, *Scientific American*
87. Walter WG (1951) A Machine that Learns, *Scientific American*
88. Wang W, Slotine J-JE (2005) On Partial Contraction Analysis for Coupled Nonlinear Oscillators. *Biological Cybernetics*, 92(1), 2005
89. Wang W, Slotine J-JE (2004) Adaptive Synchronization in Coupled Dynamic Networks, *NSL Report 040102, MIT, Cambridge, MA*, submitted
90. Wang W, Slotine J-JE (2004) Where To Go and How To Go: a Theoretical Study of Different Leader Roles, *NSL Report 040202, MIT, Cambridge, MA*, submitted
91. Wang W, Slotine J-JE (2004) Contraction Analysis of Time-Delayed Communications Using Simplified Wave Variables
92. Wang W, Slotine J-JE (2005) Fast Computation with Neural Oscillators. *Neurocomputing* 63(5)
93. Wiener N (1961) Cybernetics, *M.I.T. Press*
94. Won J, Hogan N (1995) Stability properties of human reaching movements, *Exp Brain Res. 107(1)*
95. Yazdanbakhsh A, Grossberg S (2004) Fast Synchronization of Perceptual Grouping in Laminar Visual Cortical Circuits. *Neural Networks* 17

96. Zhang Y, et al. (2001) Partial Synchronization and Spontaneous Spatial Ordering in Coupled Chaotic Systems. *Phys. Rev. E*
97. Zhao Y, Slotine J-JE (2005) Discrete Nonlinear Observers for Inertial Navigation, *Systems and Control Letters*