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Nonconvexity of the Stability Domain of Digital Filters

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Abstract—The denominator of a rational digital filter of n th order is a polynomial represented by a point A of the n -dimensional space of its coefficients. The set of all points A corresponding to stable filters defines a region Σ in this space called the *stability domain*. It is shown that for $n \geq 3$, Σ is not convex and especially if point A belongs to Σ and α satisfies $0 < \alpha \leq 1$, then point αA does not necessarily belong to Σ .

I. INTRODUCTION

The stability of a causal digital filter with a rational transfer function $H(z)$ is completely specified by the location of its poles with respect to the unit circle (UC) [1]. These poles are the roots of the polynomial appearing in the denominator which can be written as

$$P(z) = \sum_{i=0}^n a_i z^{n-i}. \quad (1)$$

As it is always possible to assume that $a_0 = 1$, the polynomial is defined by n coefficients a_i , $1 \leq i \leq n$, or by the vector $\mathbf{a} = [a_1, \dots, a_n]^T$. There is a one-to-one correspondence between vector $\mathbf{a}$ and the point A with coordinates $(a_1, \dots, a_n)$.

The set of points A corresponding to stable filters, i.e., to polynomials P with roots inside the UC, is called the *stability domain* of digital filters and denoted by Σ . It is clear that the vector $\mathbf{a} = \mathbf{0}$ corresponds to z^n and thus defines a stable filter, which means that the point O is in the stability domain.

A better understanding of the geometry and the topology of Σ is useful in the digital filter design [2]–[5]. Consider an adaptive purely recursive filter for which vector $\mathbf{a}$ varies in time. For testing

stability [6], we can, for example, compute the reflection coefficients from vector $\mathbf{a}$, which requires $O(n^2/2)$ numerical operations at each time. In addition, these coefficients can be updated from time sample to time sample in an adaptive filtering context [7]. But it is clear that this procedure is slow compared to some sufficient stability conditions on $\mathbf{a}$ given in [8] and directly related to the geometry of Σ . In the design of linear predictive coding digital filters, the constraints involved and Σ are represented [5] by sets in the same n -dimensional space of the filter coefficients and topological properties of these sets are useful.

Unfortunately, the structure of Σ is complicated for $n \geq 3$, even in the real case. An approach to its study is given for $n = 3$ in [3] and some general results are discussed in [5] and [8].

This correspondence establishes results that concern the convexity of the stability domain Σ . A region $\mathcal{R}$ of an n -dimensional space is said to be *convex* if it satisfies

$$\forall (\mathbf{a}, \mathbf{b}) \in \mathcal{R}^2, \quad \text{then } \alpha \mathbf{a} + (1 - \alpha) \mathbf{b} \in \mathcal{R}, \quad \forall \alpha \in [0, 1]. \quad (2)$$

In the real case and for $n = 2$, it is well known that Σ is the interior of a triangle [1], [2]. As a consequence, Σ is convex and, in particular, if A belongs to Σ , then all the points αA , $0 \leq \alpha \leq 1$, also belong to Σ . Unfortunately, we will prove in the following section that this property is not valid for $n > 2$ and, as a consequence of this result, the nonconvexity of the stability domain for $n > 2$.

Comment: Let a polynomial $P(z)$ be represented by a point R of the n -dimensional space of its roots and Σ be the stability domain in this space. If R is the point associated with the polynomials $P(z)$ having r_i , $1 \leq i \leq n$, as roots, the point αR is associated with $Q(z) = P(z/\alpha)$ having as roots αr_i , $1 \leq i \leq n$. It is obvious that if R belongs to Σ , then αR also belongs to Σ for $0 \leq \alpha \leq 1$ and $n \geq 1$. This property is to be compared to the result above.

II. FUNDAMENTAL RESULT

If the vector $\mathbf{s}$ or the point S is associated with the polynomial

$$P_1(z) = (z - 1)^n \quad (3)$$

the vector $\alpha \mathbf{s}$ or the point αS is associated with the polynomial

$$P_\alpha(z) = z^n + \alpha \{(z - 1)^n - z^n\} \quad (4)$$

for any real α . As P_1 has only roots on the UC, point S belongs to the boundary of Σ and we have the following result.

Theorem: For any $n > 2$ and α such that $1/2 < \alpha < 1$, all the points αS are outside Σ .

Proof: As point αS is associated with polynomial P_α given by (4), it is sufficient to prove that for any α such that $1/2 < \alpha < 1$, P_α has at least one root outside the UC. According to (4), each root x of P_α satisfies $x \neq 0$ and

$$\alpha(x - 1)^n = (\alpha - 1)x^n. \quad (5)$$

Thus

$$y'' = -\rho \quad (6)$$

where

$$y = 1 - 1/x \quad (7)$$

and

$$\rho = 1/\alpha - 1. \quad (8)$$

As we have $0 < \rho < 1$ for any α such that $1/2 < \alpha < 1$, the roots y_j of (6) are given by

$$y_j = \rho^{1/n} e^{i\theta_j}, \quad \theta_j = \pi/n + 2j\pi/n, \quad i^2 = -1, \quad 1 \leq j \leq n. \quad (9)$$

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Fig. 1. Position with respect to the UC of the roots of polynomial (4) for $n = 3$ and $n = 10$, when α decreases from 1 to 10^{-8} .

Therefore

$$1/x_j = 1 - \rho^{1/n} e^{i\delta_j} \quad (10)$$

and

$$1/|x_j|^2 = 1 + \rho^{1/n} \{ \rho^{1/n} - 2 \cos(\delta_j) \}. \quad (11)$$

It is sufficient to find $j_0 \in [1, n]$ such that $1/|x_{j_0}|^2 < 1$. But (9) gives

$$\delta_j - \delta_{j-1} = 2\pi/n, \quad 1 \leq j \leq n. \quad (12)$$

Thus, for $n \geq 3$, there exists at least one value j_0 of j such that $\delta_{j_0} \in [-\pi/3, \pi/3]$ and we have

$$1 \leq 2 \cos(\delta_{j_0}). \quad (13)$$

As $0 < \rho^{1/n} < 1$, we have

$$\rho^{1/n} - 2 \cos(\delta_{j_0}) < 0. \quad (14)$$

Relation (11) then yields $1/|x_{j_0}|^2 < 1$ and finally $|x_{j_0}|^2 > 1$.

Corollary: The stability domain Σ is nonconvex for $n \geq 3$.

Proof: As vector $\mathbf{0}$ belongs to Σ , the property above shows that $\alpha s + (1 - \alpha)\mathbf{0} = \alpha s$ is outside Σ for $1/2 < \alpha < 1$. This implies that the stability domain Σ is nonconvex for $n \geq 3$.

Example: Fig. 1 shows the position with respect to the UC of the roots of polynomial (4) for $n = 3$ and $n = 10$, when α decreases from 1 to 10^{-8} .

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Limit Cycles Due to Roundoff in State-Space Digital Filters

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Abstract—A deterministic analysis is presented for limit cycles due to roundoff in state-space digital filters. Two new sufficient conditions are established for the roundoff stability, i.e., suppression of limit cycles, in digital filters. These conditions are used to obtain roundoff stable digital filter realizations.

I. INTRODUCTION

When a digital filter is implemented, errors due to finite register length are unavoidable. The finite arithmetic in a computer generates overflow, truncation, or roundoff error due to quantization nonlinearities. These nonlinear effects give rise to self-sustained periodic oscillations, called limit cycles. Therefore, digital filters have to be carefully designed to suppress these undesirable oscillations.

Barnes and Fam [1] presented minimum norm digital filter structures and showed that these filters are overflow stable provided all the poles are inside the unit circle. Overflow oscillations in digital filters have also been studied in [2], [3]. However, if rounding is present, limit cycles can occur, which are substantially larger than those generated due to overflow [4].

Limit cycles due to roundoff in digital filters have also received some attention in the technical literature. In [6], second-order digital filters have been studied for limit cycles due to roundoff. In [7], several general conditions have been established for the existence of limit cycles due to roundoff in zero input digital filters.

In this paper, a deterministic study of limit cycles due to roundoff is given for n -order digital filters. Two new sufficient conditions are given for the nonexistence of limit cycles, that is, for roundoff stability in state-space digital filters (Theorems 1 and 2). Using these results, restrictions on the location of poles are found for various realizations to be free from roundoff limit cycles.

II. PROPERTIES OF THE SYSTEM MATRIX

Consider an infinite impulse response digital filter realization with the state-space description

$$x(n+1) = Ax(n) + Bu(n) \quad (1)$$

$$y(n) = Cx(n) + Du(n) \quad (2)$$

where A is $p \times p$, B is $p \times 1$, C is $1 \times p$, and D is scalar. The associated transfer matrix is given by

$$H(z) = C(zI - A)^{-1}B + D. \quad (3)$$

For a controllable and observable filter, the eigenvalues of the system matrix A are the poles of the transfer function $H(z)$. In what follows, zero-input limit cycles will be investigated and hence $u(n) = 0$. A realistic model of the system with a quantizer in the loop is shown in Fig. 1. With zero input and rounding, the filter satisfies the state equation

$$x(n+1) = [Ax(n)]_R \quad (4)$$

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