MAXIMUM ENTROPIES COPULAS

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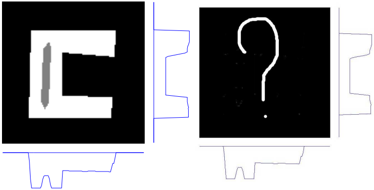
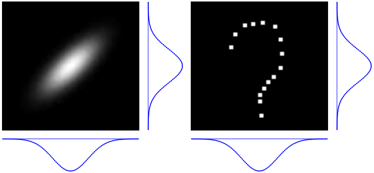
Tomography :
Given two projections horizontal and vertical $f_1(x)$ and $f_2(y)$, find image $f(x, y)$



Statistics :
Given two marginals pdfs $f_1(x)$ et $f_2(y)$, find the joint distribution $f(x, y)$



Two equivalent **ill-posed Inverse Problems** : Infinite number of solutions

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WHAT IS A COPULA ?

1. SIMPLE DEFINITION : A copula is a multivariate probability distribution function defined on $[0, 1]^n$ whose marginals are uniform.
2. IN THE STATISTICS LITERATURE
   copula is a tool to link a multivariate distribution function to its marginal distributions.

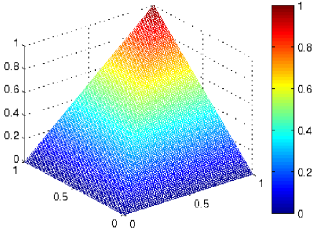
$$f(x, y) = f_1(x) f_2(y) \Omega(x, y).$$

3. POWERFUL TOOLS IN MODELING
   - Mostly used in Finance and Environmental Sciences
     (C. Genest & MacKay, 1986 ; R.M Cooke, 1997 ; P. Embrechts, 2003)
   - Offer several choices to model dependency between variables
     (H. Joe, 1997 ; R.B. Nelsen, 2006)

COPULA DEFINITION

A bivariate copula C is a function

$$\begin{array}{rcl} C : [0,1]\times[0,1] & \longrightarrow & [0,1] \\ (u,v) & \longrightarrow & C(u,v) \end{array}$$



with the following properties :

1. $C(u,0) = 0 = C(0,v)$,
2. $C(u,1) = u$ and $C(1,v) = v$,
3. $C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0$, for all $0 \leq u_1 \leq u_2 \leq 1$ and $0 \leq v_1 \leq v_2 \leq 1$,
4. $C(u,v) \leq \min\{u,v\}$,
5. $C(u,v) \geq \max\{u+v-1,0\}$.

SKLAR's THEOREM (1959)

Let F be a two-dimensional distribution function with marginal distributions functions F_1 and F_2. Then there **exists** a copula C such that :

$$F(x, y) = C(F_1(x), F_2(y)). \tag{1}$$

If the marginal functions are continuous, then the copula C is **unique**, and is given by

$$C(u, v) = F(F_1^{-1}(u), F_2^{-1}(v)). \tag{2}$$

Otherwise C is uniquely determined on $Ran\ (F_1) \times Ran\ (F_2)$.
Conversely for any univariate distribution functions F_1 and F_2 and any copula C, the function F is a two-dimensional distribution function with marginals F_1 and F_2, given by (1).

DIRECT INVERSION METHOD

$$f(x,y) = f_1(x) f_2(y) c(F_1(x), F_2(y))$$

$$c(u,v) = \frac{f\left[F_1^{-1}(u), F_2^{-1}(v)\right]}{f_1\left[F_1^{-1}(u)\right] f_2\left[F_2^{-1}(v)\right]}$$

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ELLIPTICAL COPULAS

1. **Gaussian Copula** : with one parameter ρ
   - Φ_Σ : cdf of bivariate standard Gaussian.
   - Σ : covariance matrix, with correlation coefficientρ

$$C_\rho(u,v) = \Phi_\Sigma\left(\Phi^{-1}(u), \Phi^{-1}(v)\right)$$

2. **Student Copula** with two parameters ρ and ν
   - $t_{\Sigma,\nu}$: cdf of a bivariate Student distribution,
   - Σ : covariance matrix with correlation coefficient ρ,
   - ν : the degree of freedom

$$C_{\rho,\nu}(u,v) = t_{\Sigma,\nu}\left(t_\nu^{-1}(u), t_\nu^{-1}(v)\right)$$

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ARCHIMEDEAN COPULAS

For φ a non increasing and convex function, where $\varphi(0) = \infty$, $\varphi(1) = 0$:

$$C(u, v) = \varphi^{-1}\left(\varphi(u) + \varphi(v)\right).$$

Important : Multivariate dependence is captivated by an univariate function.
Some examples :

1. **Clayton Copula** (1978) : with one non-null parameter $\alpha \in [-1, \infty)$
    - The generator $\varphi(t) = \frac{1}{\alpha}\left(t^{-\alpha} - 1\right)$

$$C(u, v; \alpha) = \left[\left[u^{-\alpha} + v^{-\alpha} - 1\right]^{\frac{-1}{\alpha}}\right]_{+}$$

2. **Gumbel Copula** (1960) : with one parameter $0 < \alpha \leq 1$.
    - The generator $\varphi(t) = \ln(1 - \alpha \ln(t))$

$$C_{\alpha}(u, v) = uv \exp\left(-\alpha \ln u \ln v\right).$$

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Maximum Entropies Copulas

Problem : Given the two marginals $f_1(x)$ and $f_2(y)$ find the joint pdf $f(x,y)$

Solution : Select the solution which maximizes an entropy

Mathematics : Maximize

- $J_1(f) = -\iint f(x,y) \ln f(x,y)\, dx\, dy,$

subject to

$$\begin{cases} C_1 : \int f(x,y)\, dy = f_1(x), \quad \forall x \\ C_2 : \int f(x,y)\, dx = f_2(y), \quad \forall y \\ C_3 : \iint f(x,y)\, dx\, dy = 1. \end{cases}$$

Differents Entropies Expressions

Rényi, Burg and Tsallis entropies

- $J_2(f) = \dfrac{1}{1-q} \ln \left(\iint f^q(x,y)\, dx\, dy \right)$, $q \geqslant 0$ and $q \neq 1$,
- $J_3(f) = \iint \ln f(x,y)\, dx\, dy$,
- $J_4(f) = \dfrac{1}{1-q} \left(1 - \iint f^q(x,y)\, dx\, dy \right)$ $q \geqslant 0$ and $q \neq 1$.

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Lagrange Multipliers Technique

$$\mathcal{L}(f,\lambda_0,\lambda_1,\lambda_2) = J_i(f) + \lambda_0\left(1 - \iint f(x,y)dxdy\right) + \int \lambda_1(x)\left(f_1(x) - \int f(x,y)dy\right)dx + \int \lambda_2(y)\left(f_2(y) - \int f(x,y)dx\right)dy,$$

Solution

$$\begin{cases} \partial\mathcal{L}/\partial f = 0 \\ \partial\mathcal{L}/\partial\lambda_0 = 0 \\ \partial\mathcal{L}/\partial\lambda_1 = 0 \\ \partial\mathcal{L}/\partial\lambda_2 = 0. \end{cases}$$

Shannon's Entropy

$$f(x,y) = f_1(x)f_2(y)$$

$$f(x_1,\ldots,x_n) = \prod_{i=1}^{n} f_i(x_i)$$

$$F(x_1,\ldots,x_n) = \int_0^{x_1} \ldots \int_0^{x_n} \prod_{i=1}^{n} f_i(s_i) \prod_{i=1}^{n} ds_i, \quad 0 \leq x_i \leq 1.$$

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TSALLIS' ENTROPY INDEX $q = 2$

Pdf :

$$f(x, y) = [f_1(x) + f_2(y) - 1]_+ .$$

$$f(x_1, \dots, x_n) = \left[\sum_{i=1}^{n} f_i(x_i) - n + 1 \right]_+$$

Cdf :

$$F(x, y) = \int_0^x \int_0^y f(s, t)\, ds\, dt$$

$$F(x, y) = y\, F_1(x) + x\, F_2(y) - x\, y, \qquad 0 \leq x, y \leq 1.$$

$$F(x_1, \dots, x_n) = \sum_{i=1}^{n} F_i(x_i) \prod_{\substack{j=1 \\ j \neq i}}^{n} x_j + (1 - n) \prod_{i=1}^{n} x_i, \qquad 0 \leq x_i \leq 1.$$

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Direct Inversion Method

Given the expressions of $F(x,y)$ and $F_1(x)$, $F_2(y)$, the expressions of copula becomes :

$$C(u,v) = \left[u\,F_2^{-1}(v) + v\,F_1^{-1}(u) - F_1^{-1}(u)\,F_2^{-1}(v)\right]_+$$

Multivariate case :

$$C(u_1,\ldots,u_n) = \left[\sum_{i=1}^{n} u_i \prod_{\substack{j=1 \\ j\neq i}}^{n} F_j^{-1}(u_j) + (1-n)\prod_{i=1}^{n} F_i^{-1}(u_i)\right]_+$$

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Beta Distribution Marginals

One great family of distributions defined on $[0,1]$:

$$f_1(x) = \frac{1}{B(a_1,b_1)} x^{a_1-1}(1-x)^{b_1-1}$$

$$f_2(y) = \frac{1}{B(a_2,b_2)} y^{a_2-1}(1-y)^{b_2-1}\,,$$

where

$$B(a_i,b_j) = \int_0^1 t^{a_i-1}(1-t)^{b_j-1}\,dt,\ 0 \leq x, y \leq 1 \,\text{and}\; a_i, b_j > 0\,.$$

Interesting particular cases :

case 1 : $a_i > 0\,,\ b_j = 1$

case 2 : $a_i = b_j = 1/2$

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New Families of copulas

case 1 : $a_i > 0\,,\, b_j = 1$

$$\begin{cases} f_1(x) = a_1 x^{a_1-1} \to F_1(x) = x^{a_1} \to F_1^{-1}(u) = u^{\frac{1}{a_1}} \\ f_2(y) = a_2 y^{a_2-1} \to F_2(y) = y^{a_2} \to F_2^{-1}(v) = v^{\frac{1}{a_2}} \end{cases}$$

$$F(x,y;a_1,a_2) = y\,x^{a_1} + x\,y^{a_2} - x\,y, \quad 0 \le x,y \le 1.$$

The corresponding copula :

$$C(u,v;a_1,a_2) = u\,v^{\frac{1}{a_2}} + v\,u^{\frac{1}{a_1}} - u^{\frac{1}{a_1}}\,v^{\frac{1}{a_2}}$$

is well defined for appropriate values of a_1, a_2 and for almost u, v in $[0,1]$.

If $a_1 = a_2 = \dfrac{1}{a}$,

$$C(u,v;a) = (u\,v)^a (u^{1-a} \otimes_1 v^{1-a})$$

where $u \otimes_a v = [u^a + v^a - 1]^{\frac{1}{a}}$ is the generalized product.

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New Families of copulas

Case 2 : $a_i = b_j = 1/2$

$$\left\{ \begin{array}{l} f_1(x) = \dfrac{1}{\pi\sqrt{x(1-x)}} \to F_1(x) = \dfrac{2}{\pi}\arcsin(\sqrt{x}) \to F_1^{-1}(u) = \sin^2(\frac{\pi}{2}u) \\ f_2(y) = \dfrac{1}{\pi\sqrt{y(1-y)}} \to F_2(y) = \dfrac{2}{\pi}\arcsin(\sqrt{y}) \to F_2^{-1}(v) = \sin^2(\frac{\pi}{2}v) \end{array} \right\}$$

$$F(x,y) = \frac{2y}{\pi}\arcsin(\sqrt{x}) + \frac{2x}{\pi}\arcsin(\sqrt{y}) - x\,y, \quad 0 \le x, y \le 1.$$

The corresponding copula :

$$C(u,v) = u\,\sin^2(\frac{\pi v}{2}) + v\,\sin^2(\frac{\pi u}{2}) - \sin^2(\frac{\pi u}{2})\sin^2(\frac{\pi v}{2})$$

is well defined for all u, v in $[0,1]$.

CONCLUSION

- New families of copula, other examples are in investigation
- Link betwen copula & Tomography
  Shannon : $f(x, y) = f_1(x) f_2(y)$:   Multiplicative Backprojection
  Tsallis : $f(x, y) = [f_1(x) + f_2(y) - 1]_+$: Backprojection Method

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