On escort distributions, $q$-gaussians and Fisher information

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## Outline

1. **Escort distributions**
   - Escort distribution
   - Applications - Quantization & Source coding

2. **Rényi-Tsallis and $q$-gaussians**
   - Rényi-Tsallis and Co
   - The escort path

3. **Minimum Fisher**
   - Fisher min with $q$-variance constraint
   - Equivalent pseudo-convex problem
   - Solutions and Cramér-Rao inequalities

4. **Results**
   - Numerical results

5. **TheEnd**
Escort distributions originate from multifractals.

Actually the singular spectrum $f(\alpha)$ and the Rényi dimension are linked by a Legendre transform.

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A related MaxEnt problem is

$$f(\alpha) = \max_P H_1(P) = -\sum P_i \log P_i$$

subject to $\alpha = \sum P_i \log p_i$ and normalization, that leads to

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Escort distribution

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- Actually the singular spectrum $f(\alpha)$ and the Rényi dimension are linked by a Legendre transform.
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subject to $\alpha = \sum P_i \log p_i$ and normalization, that leads to

$$P_i = \frac{p_i^q}{\sum p_i^q}$$

It can be shown that

$$D(U||P) \geq D(U||p) \text{ if } q > 1$$

$$D(U||P) \leq D(U||p) \text{ if } q < 1$$

Applications - Quantization

In a nonuniform companding quantization

Distorsion

\[ D = \sum_i \int_{x_i}^{x_i+1} (x - y_i)^p f_X(x) \, dx \]

in the HR limit, the density of points that minimizes \( D \) is

\[ \lambda(x) = \frac{f_X(x)^q}{\int f_X(x)^q \, dx} \propto G'(x) \quad \text{with} \quad q = \frac{1}{p+1}. \]

- Possible trade-off between entropic quantization \( \lambda(x) = cte \) and density distorsion (work in progress).


And also source coding...

- 1948, Shannon’s source coding theorem
- en 1965, L.L. Campbell the first *operational* characterization of Rényi entropy...

**Escort-distribution**: \( P_i = \frac{p_i^q}{\sum_i p_i^q} \)

<table>
<thead>
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<th>length</th>
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<th>opt. length</th>
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<tr>
<td>Shannon</td>
<td>( \bar{L} = \sum_i p_i l_i )</td>
<td>( H_1(p) )</td>
<td>( l_i = - \log_D(p_i) )</td>
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<tr>
<td>Campbell</td>
<td>( C_\beta = \frac{1}{\beta} \log_D \sum_{i=1}^N p_i D^{\beta l_i} )</td>
<td>( H_{1/(\beta+1)}(p) )</td>
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<tr>
<td>Gen. mean</td>
<td>( M_q = \sum_{i=1}^N P_i l_i )</td>
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- Generalized mean: Obvious way to obtain the Campbell’s codes using a standard coder!

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Rényi-Tsallis entropy maximization

**Proposition**

The probability distribution that maximizes the Tsallis entropy

\[ H_q[f] = \frac{1}{q-1} \left( 1 - \int f(x)^q \, dx \right) \]

subject to the \( q \)-variance constraint \( \sigma_q^2 = \frac{\int x^2 f(x)^q \, dx}{\int f(x)^q \, dx} \) is the \( q \)-gaussian distribution defined by

\[ p_q(x) = \frac{1}{Z_q(\beta)} \left( 1 - (1 - q)\beta x^2 \right)^{\frac{1}{1-q}}, \quad q \neq 1 \]
Rényi-Tsallis entropy maximization

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\[ \rho_q(x) = \frac{1}{Z_q(\beta)} \left( 1 - (1-q)\beta x^2 \right)^{\frac{1}{1-q}}, \quad q \neq 1 \]

Proof: Compute the Rényi information divergence \( D_q(p||\rho_q) \) with \( p \) admissible, use the \( q \)-variance constraint and check that

\[ D_q(p||\rho_q) = H_q(\rho_q) - H_q(p) \geq 0. \]
qx-gaussians

Example of qx-gaussians for \( q \leq 1 \)
$q$-gaussians

Example of $q$-gaussians for $q \geq 1$
The escort path

The problem

\[ \text{Min}_p D(p\|p_1) \text{ s.t. } D(p\|p_0) = \eta \]
The escort path

- The problem

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[Some similarities with Carlos’ setting]
The escort path

The problem

\[
\text{Min}_p D(p||p_1) \text{ s.t. } D(p||p_0) = \eta
\]

leads to

\[
p_q = p_0^{1-q} p_1^q / N_q \text{ with } N_q = \int p_0^{1-q} p_1^q.
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leads to \( p_q = p_0^{1-q} p_1^q / N_q \) with \( N_q = \int p_0^{1-q} p_1^q \).

\( p_q \) describe a path from \( p_0 \) (\( q = 0 \)) to \( p_1 \) (\( q = 1 \)).
The escort path

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\[ \min_p D(p \| p_1) \text{ s.t. } D(p \| p_0) = \eta \]

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\( p_q \) describe a path from \( p_0 \) (\( q = 0 \)) to \( p_1 \) (\( q = 1 \)).

\[ D(p_q \| p_1) = A + B \cdot D_q(p_1 \| p_0) \]
Selecting f...

On the setting $U \leftrightarrow f_q = \frac{f_q}{\int f_q} \leftrightarrow f$,

How to select $f/f_q$?

- If $\sigma^2_q = E_{f_q}[X^2]$ is fixed?
- Or if $\sigma^2 = E_f[X^2]$ is fixed?
Selecting f...

On the setting $U \leftrightarrow f_q = \frac{f_q}{\int f_q} \leftrightarrow f$,

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And what about the evolution of such optimum distributions when $q$ varies?
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And what about the evolution of such optimum distributions when $q$ varies?

We know that $\max_f H_q[f]$ s.t variance constraint $\Rightarrow$ a $q$-gaussian.
Selecting f - Fisher minimization...

- Fisher minimization
  - We use the Fisher information of the distribution (translation families)
  - important applications in statistical physics, [Frieden et al]
  - information theoretic grounds

- Paths with minimum length – thermodynamic length/divergence [Weinhold-Crooks]= flux of Fisher information along the curve

- What about the most general distribution in the Fisher sense?

First problem: distributions with fixed $q$-variance

$$\inf_{f} \int f(x) \left( \frac{f'(x)}{f(x)} \right)^2 dx = I[f],$$

$$s.t. \quad \sigma_q^2 = \frac{\int x^2 f(x)^q dx}{\int f(x)^q dx}$$

On escort distributions, $q$-gaussians and Fisher information
Fisher min with $q$-variance constraint

- Even if $I[f]$ convex, the constraint is not and uniqueness can not be guaranteed
- The formulation of the constraint is awkward

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\begin{align*}
\inf_f I[f], \\
\text{s.t. } \sigma_q^2 &= \frac{\int x^2 f(x)^q dx}{\int f(x)^q dx}, \\
\text{s.t. } f(x) &\geq 0, \int f(x) dx = 1,
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\]

\[
\inf_{f \geq 0} I[f], \quad \text{s.t.} \quad N_q = \int f(x)^q dx \quad \text{and} \quad V_q = \sigma_q^2 N_q = \int x^2 f(x)^q dx
\]

- The initial constraint still present since $\sigma_q^2 = V_q / N_q$. 

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- A two steps procedure
Fisher min with \( q \)-variance constraint

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- The initial constraint still present since \( \sigma^2_q = V_q / N_q \).
- A two steps procedure.
- Expect a parametric solution on positive functions; the normalized version is the solution on the subset \( \int f(x) dx = 1 \).
A simple transformation

- Nonlinear equality constraints into linear equality constraints...
- $f_q(x) = f(x)^q \rightarrow$ substitute $f(x)^q$ by $f_q(x)$
A simple transformation

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- $f_q(x) = f(x)^q \rightarrow$ substitute $f(x)^q$ by $f_q(x)$
- and the whole problem

$$\begin{cases} \inf_f I[f], \\ s.t. \quad N_q = \int f(x)^q \, dx \\ and \quad V_q = \sigma_q^2 N_q = \int x^2 f(x)^q \, dx \end{cases}$$
A simple transformation

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- $f_q(x) = f(x)^q \rightarrow$ substitute $f(x)^q$ by $f_q(x)$
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s.t. \quad N_q = \int f_q(x)dx, \\
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\[
\left\{ \begin{array}{l}
\inf_{f_q} l_{\bar{q}} [f_q], \\
s.t. \quad N_q = \int f_q(x) dx, \\
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\end{array} \right.
\]

with, for the Fisher information

\[
l[f] = l \left[ f_{\bar{q}} \right] = \bar{q}^2 \int f_q(x) \bar{q} \left( \frac{f_q'(x)}{f_q(x)} \right)^2 \, dx = l_{\bar{q}} [f_q]. \quad \bar{q} = 1/q
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I[f] = I \left[ f_q^\bar{q} \right] = \bar{q}^2 \int f_q(x) \bar{q} \left( \frac{f_q'(x)}{f_q(x)} \right)^2 dx = I_q \left[ f_q \right]. \quad \bar{q} = 1/q
\]

- Since we have reasons to suspect it... we check that...
A simple transformation

- Nonlinear equality constraints into linear equality constraints...
- \( f_q(x) = f(x)^q \) → substitute \( f(x)^q \) by \( f_q(x) \)
- and the whole problem becomes

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\end{aligned}
\]

with, for the Fisher information

\[
l[f] = I[\tilde{f}_q] = \tilde{q}^2 \int f_q(x) \frac{f'_q(x)}{f_q(x)}^2 dx = I_{\bar{q}}[f_q]. \quad \tilde{q} = 1/q
\]

- Since we have reasons to suspect it... we check that

\[\rightarrow\] The transformation of the \( q \)-gaussian with \( f_q(x) = f(x)^q \) satisfies the new Euler-Lagrange equation. Minimum? Global?
A pseudo-convex formulation

- The $\bar{q}$-Fisher information $I_{\bar{q}}[f_q] = I[f_{\bar{q}}]$ is the composition of a convex function $I[.]$ and of the monotone function $f_{\bar{q}}$.
- Intuitively, $I_{\bar{q}}[.]$ is either monotone or unimodal - its minimization subject to linear constraints shall lead to a unique solution.

Definition

A function $f(x)$ defined on an open set $\Gamma$ is said pseudo-convex at $x_0$ if it is differentiable at $x_0$ and

$$f(x) - f(x_0) < 0 \Rightarrow f'(x_0)(x - x_0) < 0 \quad \forall x \in \Gamma.$$ 

If this is true for all $x_0 \in \Gamma$, the function is said pseudo-convex on $\Gamma$ [Mangasarian 1987]
A pseudo-convex formulation

- The $\bar{q}$-Fisher information $l_{\bar{q}}[f_q] = l \left[ f_{\bar{q}} \right]$ is the composition of a convex function $l[.]$ and of the monotone function $f_{\bar{q}}$.
- Intuitively, $l_{\bar{q}}[.]$ is either monotone or unimodal - its minimization subject to linear constraints shall lead to a unique solution.

Example –
Can be extended to the functional case \(\rightarrow\) Pseudo convex functional
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Result

If $H[p] = \int h(p(x)) \, dx$ where $h$ is the composition of a strictly convex function and a monotone function, then $H[p]$ is pseudo-convex.
Can be extended to the functional case \( \rightarrow \) Pseudo convex functional

**Result**

If \( H[p] = \int h(p(x)) \, dx \) where \( h \) is the composition of a strictly convex function and a monotone function, then \( H[p] \) is pseudo-convex.

**Result**

If \( H[p] = \int h(p(x)) \, dx \) is a pseudo-convex functional and if \( p_0(x) \) is an admissible stationary point of the Lagrangian associated with

\[
\min_{p \in \Gamma_p} H[p] \text{ subject to } \int a_i(x)p(x) \, dx = c_i, \; i = 1 \ldots n.
\]

then \( p_0(x) \) is a global minimum.
So we get finally the result:

**Theorem**

For $q \in [0,5)$, the probability density function that minimizes the Fisher information under q-variance constraint is the q-gaussian distribution

$$f_*(x) = \frac{1}{Z_q(\beta)} \left( 1 - (1 - q)\beta x^2 \right)^{\frac{2}{1-q}} , \quad q \neq 1$$

where the q-variance is given by

$$\sigma_q^2 = \frac{1}{\beta (q + 3)}.$$
Generalized Cramér-Rao inequality

- For any distribution $f(x)$ with the same $q$-variance as the $q$-gaussian $f^*(x)$ we have $I[f] \geq I[f^*]$.

- We can derive the expression of $I[f^*]$ in term of $q$ and the $q$-variance. This leads us to the following inequality:
Generalized Cramér-Rao inequality

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**Corollary**

*(Cramér-Rao inequality).* For any distribution \( f \) with \( q \)-variance \( \sigma_q^2 \), we have

\[
I[f] \geq I[f^*] = \frac{2(5-q)}{(q+1)(q+3)} \frac{1}{\sigma_q^2},
\]

with equality if \( f \) is the \( q \)-gaussian \( f^* \).
Result on Escort-Fisher

Let

\[ p(x) = \frac{f(x)^q}{\int f(x)^q \, dx} \quad f(x) = \frac{p(x)^{\bar{q}}}{\int p(x)^{\bar{q}} \, dx}. \]

Then, we obtain an Escort-Fisher of order \( \bar{q} \)

\[ I[f] = I_{\bar{q}}[p] = \bar{q}^2 \int \frac{p(x)^{\bar{q}}}{\int p(x)^{\bar{q}} \, dx} \left( \frac{p'(x)}{p(x)} \right)^2 \, dx \]

and

\[
\begin{align*}
\left\{ \begin{array}{c}
\inf_f I[f], \\
n. t. \quad \sigma_q^2 = \frac{\int x^2 f(x)^q \, dx}{\int f(x)^q \, dx} \\
\text{and} \quad f \geq 0 \quad \int f(x) \, dx = 1
\end{array} \right. \\
= \left\{ \begin{array}{c}
\inf_p I_{\bar{q}}[p], \\
n. t. \quad \sigma_q^2 = \int x^2 p(x) \, dx \\
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\end{array} \right.
\]

- New problem with standard variance constraint.
- Same global minimum value and \( p_* \) is the escort of order \( q \) of \( f_* \).
If we swap $q$ and $\bar{q} = 1/q$, a similar result follows

**Theorem**

For $q > \frac{1}{5}$, the probability density function that minimizes the Escort-Fisher information of order $q$ under a standard variance constraint is the $q$-gaussian distribution

$$p_*(x) = \frac{1}{Z_{\bar{q}}(\beta)} \left( 1 - \frac{(q - 1)}{q} \beta x^2 \right)^{\frac{2}{q-1}}.$$

**Corollary**

*(Cramér-Rao inequality for escort-Fisher). For any distribution $p$ with a given (standard) variance $\sigma^2$, the escort-Fisher information of order $q > \frac{1}{5}$ satisfies*

$$I_q[p] \geq \frac{2q(5q - 1)}{(q + 1)(3q + 1)} \frac{1}{\sigma^2}.$$
Numerical results

- We use a parametric model of probability densities
- Transform the variational problems into optimizations over (a finite number of) parameters.
- We work with the model of densities:

\[ p(x) = u(x)^2 = \frac{A}{\left( \sum_{k=1}^{p} a_k x^k \right)^2}. \]

- Rationale: Weierstrass approximation theorem,
- Implementation of the variational Fisher minimization problems achieved using the active-set algorithm provided in the fmincon script of the Matlab software.
- Initial condition: coefficients of the polynomial that fits the inverse of a normal density on the interval I. Order \( p = 10, \)
Comparison of numerical/analytical results

Figure: Comparison of minimum Fisher information distributions (plain lines) and $q$-gaussians (dashed lines) for several values of $q$. For values of $q < 1$, the distributions have compact support, and heavy tails for $q > 1$. The comparison shows a remarkable agreement of experimental results with analytical derivations.
Cramér-Rao planes

- We randomly generate densities $f$ according to the model.
- Compute $I[f]$, $I_q[f]$, $\sigma^2$, $\sigma_q^2$ and populate the plane with the variance-Fisher coordinates.
Cramér-Rao planes

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- Compute \( I[f] \), \( I_q[f] \), \( \sigma^2 \), \( \sigma_q^2 \) and populate the plane with the variance-Fisher coordinates.

Figure: Cramér-Rao \( (\sigma_q^2, I[p]) \) information plane, with \( q = 1.2 \),
Figure: Cramér-Rao ($\sigma^2, I_q[p]$) information plane, with $q = 0.5$, populated with the coordinates of randomly generated densities $\rho$. The analytical bound in dashed line is satisfied.
Minimum Fisher Transition with constant $\sigma_q^2$

From $q = 0$ to $q = 1$, the minimum Fisher transition is obtained for $q$-gaussians with $\sigma_q^2 = 1/\beta(3 + q)$ and the thermodynamic divergence $\int_0^1 I[f_\ast]dq$ is $8 \times \ln(3) - 10 \times \ln(2) \approx 1.86$
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![Graph showing minimum Fisher transition with constant $\sigma_q^2$.](image)
Minimum escort-Fisher Transition with constant $\sigma^2$

From $q = 0$ to $q = 1$, the minimum Fisher transition is obtained for $q$-gaussians with $\sigma^2 = 1/\beta (3 + 1/q)$ and the thermodynamic divergence $\int_{1/5}^{1} l_q[p_*]dq$ is $\approx 0.42$.
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From $q = 0$ to $q = 1$, the minimum Fisher transition is obtained for $q$- gaussians with $\sigma^2 = 1/\beta(3 + 1/q)$ and the thermodynamic divergence $\int_{1/5}^{1} l_q[p_*] dq$ is $\approx 0.42$
Minimum escort-Fisher Transition with constant $\sigma^2$

From $q = 0$ to $q = 1$, the minimum Fisher transition is obtained for $q$-gaussians with $\sigma^2 = 1/\beta (3 + 1/q)$ and the thermodynamic divergence $\int_{1/5}^1 l_q[p_\ast]dq$ is $\approx 0.42$. 

![Diagram showing transition from $q=0$ to $q=1$ for $q$-gaussians with a constant $\sigma^2$.]
Conclusions and future directions

- Alternative information measures
- Role of escort distributions - Quantization, coding...
- Transition between states - Escort path
- Minimum Fisher
  - A new characterization of $q$-gaussians
  - Valid Legendre structure
  - Generalized Cramér-Rao planes $\rightarrow$ new tools
- General moments instead of $q$-variance
- Multivariate case.
- Numerical results. New inequalities?