Determination and Estimation of Generalized Entropy Rates for Markov Chains

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Shannon entropy rate of a stochastic process

- The **entropy up to time** $n$ of a random sequence $\mathbf{X} = (X_n)_{n \in \mathbb{N}}$ with denumerable state space $E$ is
  \[ - \sum_{i_1, \ldots, i_n \in E} p_n(i_1^n) \log p_n(i_1^n), \]
  where $p_n(i_1^n) = \mathbb{P}[(X_1, \ldots, X_n) = (i_1, \ldots, i_n)]$ is the likelihood of the sequence.

- The **entropy rate** of $\mathbf{X}$ is defined by
  \[ -\frac{1}{n} \sum_{i_1, \ldots, i_n \in E} p_n(i_1^n) \log p_n(i_1^n) \longrightarrow \mathbb{H}(\mathbf{X}), \quad n \rightarrow +\infty, \]
  when this quantity is finite.

- **Asymptotic Equirepartition Property**:
  \[ -\frac{1}{n} \log p_n(X_1^n) \longrightarrow \mathbb{H}(\mathbf{X}), \quad n \rightarrow +\infty, \]
  weak if the convergence is in probability,
  strong if it holds almost surely.
Generalized entropy functionals

The \((h, \phi)\)-entropy of any measure \(\nu\) on \(E\) is defined by

\[
S_{h(y), \phi(x)}(\nu) = h \left[ \sum_{i \in E} \phi(\nu(i)) \right]
\]

if \(\sum_{i \in E} \phi(\nu(i))\) is finite, and as \(+\infty\) either.

The functions \(h : \mathbb{R} \to \mathbb{R}\) and \(\phi : [0, 1] \to \mathbb{R}_+\) are twice continuously differentiable functions, with either \(\phi\) concave and \(h\) increasing or \(\phi\) convex and \(h\) decreasing.

Some \((h, \phi)\)-entropies:

<table>
<thead>
<tr>
<th>(h(y))</th>
<th>(\phi(x))</th>
<th>((h, \phi)) – entropies</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>(-x \log x)</td>
<td>Shannon (1948)</td>
</tr>
<tr>
<td>((1 - s)^{-1} \log y)</td>
<td>(x^s)</td>
<td>Renyi (1961)</td>
</tr>
<tr>
<td>([t(t - r)]^{-1} \log y)</td>
<td>(x^{r/t})</td>
<td>Varma (1966)</td>
</tr>
<tr>
<td>(y)</td>
<td>((1 - 2^{1-s})^{-1}(x - x^s))</td>
<td>Havrda and Charvat (1967)</td>
</tr>
<tr>
<td>((t - 1)^{-1}(y^t - 1))</td>
<td>(x^{1/t})</td>
<td>Arimoto (1971)</td>
</tr>
<tr>
<td>((r - 1)^{-1}[y^{(r-1)/(s-1)} - 1])</td>
<td>(x^s)</td>
<td>Sharma and Mittal 1 (1975)</td>
</tr>
<tr>
<td>((r - 1)^{-1}[\exp(r - 1)y - 1])</td>
<td>(-x \log x)</td>
<td>Sharma and Mittal 2 (1975)</td>
</tr>
<tr>
<td>(y)</td>
<td>(-x^s \log x)</td>
<td>Taneja (1975)</td>
</tr>
<tr>
<td>((r - 1)^{-1}(1 - y))</td>
<td>((t - r)^{-1}(x^r - x^t))</td>
<td>Sharma and Taneja (1975)</td>
</tr>
<tr>
<td></td>
<td>(x^r)</td>
<td>Tsallis (1988)</td>
</tr>
</tbody>
</table>

- The \((h, \phi)\)-entropy rate of a random sequence \(X = (X_n)_{n \in \mathbb{N}}\) with state space \(E \subset \mathbb{N}\) is defined by

\[
\frac{1}{n} S_{h(y), \phi(x)}(p_n) \longrightarrow \mathbb{H}_{h, \phi}(X), \quad n \to +\infty.
\]

where \(p_n(i^n_0) = \mathbb{P}(X_0 = i_0, \ldots, X_{n-1} = i_{n-1})\) is the distribution of \((X_0, \ldots, X_{n-1})\).
Quasi-power property The process $X$ satisfies the quasi-power property with parameters $[\sigma_0, \lambda, c, \rho]$ if:

1. $\sup_{i_0^n \in E^{n+1}} p_n(i_0^n) \longrightarrow 0$ when $n \to \infty$.

2. $\exists \sigma_0 \in ]-\infty, 1]$ such that $\forall s > \sigma_0$ and $\forall n \in \mathbb{N}$, the series

$$\Lambda_n(s) = \sum_{i_0^n \in E^{n+1}} p_n(i_0^n)^s$$

is convergent and satisfies

$$\Lambda_n(s) = c(s) \cdot \lambda(s)^n + R_n(s),$$

with $|R_n(s)| = O(\rho(s)^n \lambda(s)^n)$, where: $c$ and $\lambda$ are strictly positive analytic functions for $s > \sigma_0$; $\lambda$ is strictly decreasing with $\lambda(1) = c(1) = 1$, $R_n$ is also analytic, $\rho(s) < 1$.

Remarks:

The quasi-power property says that $\Lambda_n(s)$ behaves like the $n$-th power of some analytic function.

In dynamical systems theory, $\Lambda_n(s)$ is called the Dirichlet series of fundamental measures of depth $n + 1$. 
Classical entropy rates of a random sequence satisfying the quasi-power property.

<table>
<thead>
<tr>
<th>Entropy</th>
<th>Parameters</th>
<th>Entropy rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shannon</td>
<td></td>
<td>$-\lambda'(1)$</td>
</tr>
<tr>
<td>Rényi</td>
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</tr>
<tr>
<td></td>
<td>$s \neq 1$</td>
<td>$\frac{1}{1-s} \log \lambda(s)$</td>
</tr>
<tr>
<td>Varma</td>
<td>$r = t$</td>
<td>$-\frac{1}{m^2} \lambda'(1)$</td>
</tr>
<tr>
<td></td>
<td>$r \neq t$</td>
<td>$\frac{1}{t(t-r)} \log \lambda(r/t)$</td>
</tr>
<tr>
<td>Havrda-Charvat</td>
<td>$s &gt; 1$</td>
<td>0</td>
</tr>
<tr>
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<td>$-\frac{1}{\log 2} \lambda'(1)$</td>
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<tr>
<td></td>
<td>$s &lt; 1$</td>
<td>$+\infty$</td>
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<tr>
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<tr>
<td></td>
<td>$t = 1$</td>
<td>$-\lambda'(1)$</td>
</tr>
<tr>
<td></td>
<td>$t &lt; 1$</td>
<td>0</td>
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<tr>
<td>Sharma-Mittal 1</td>
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</tr>
<tr>
<td></td>
<td>$r &gt; 1$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$s = r = 1$</td>
<td>$-\lambda'(1)$</td>
</tr>
<tr>
<td></td>
<td>$r = 1 \neq s$</td>
<td>$\frac{1}{1-s} \log \lambda(s)$</td>
</tr>
<tr>
<td>Sharma-Mittal 2</td>
<td></td>
<td>$(1-s)^{-1}[\exp(-(s-1)\lambda'(1)) - 1]$</td>
</tr>
<tr>
<td>Taneja</td>
<td>$r &lt; 1$</td>
<td>$+\infty$</td>
</tr>
<tr>
<td></td>
<td>$r = 1$</td>
<td>$-\lambda'(1)$</td>
</tr>
<tr>
<td></td>
<td>$r &gt; 1$</td>
<td>0</td>
</tr>
<tr>
<td>Sharma-Taneja</td>
<td>$r &lt; 1$ or $s &lt; 1$</td>
<td>$+\infty$</td>
</tr>
<tr>
<td></td>
<td>$r &gt; 1$ and $s &gt; 1$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$r = 1$ and $s &gt; 1$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$r = 1$ and $s = 1$</td>
<td>$-\lambda'(1)$</td>
</tr>
<tr>
<td></td>
<td>$r &gt; 1$ and $s = 1$</td>
<td>0</td>
</tr>
<tr>
<td>Tsallis</td>
<td>$r &lt; 1$</td>
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</tr>
<tr>
<td></td>
<td>$r = 1$</td>
<td>$-\lambda'(1)$</td>
</tr>
<tr>
<td></td>
<td>$r &gt; 1$</td>
<td>0</td>
</tr>
</tbody>
</table>
For an i.i.d. sequence with common distribution $\nu$

Since $p_n(i_0, i_1, \ldots, i_n) = \nu(i_0)\nu(i_1)\ldots\nu(i_n)$, the Dirichlet series $\Lambda_n(s)$ can simply be written

$$\Lambda_n(s) = \left[ \sum_{i \in E} \nu(i)^s \right]^{n+1}.$$  

Hence, $X$ satisfies the quasi-power property for $s > 0$ with functions $\lambda$, $c$ and $\rho$ defined by

$$\lambda(s) = \sum_{i \in E} \nu(i)^s, \quad c(s) = 1 \quad \text{and} \quad \rho(s) = 0.$$  

For a finite chain

$\Lambda_n(s) = 1 \cdot P^n_s \cdot \nu_s$, where $P_s = (p(i, j)^s)_{i,j \in E}$, with $\nu$ the initial distribution of the chain, and $\nu_s = (\nu(i)^s)_{i \in E}$.

The following relation defines the functions $\lambda$, $c$ and $\rho$ of the quasi-power property:

$$P^n_s \cdot v = \lambda(s)^n \cdot < v, r_s > l_s + R^n(s) \cdot v,$$

where $\lambda(s)$ is the unique dominant eigenvalue of $P_s$ with maximum modulus, with associated left and right eigenvectors $l_s$ and $r_s$. 
For a denumerable chain

**Theorem Ciuperca, Girardin, Lhote (2010)**

Let $\mathbf{X} = (X_n)$ be an ergodic Markov chain with transition matrix $P$ and initial distribution $\nu$. Suppose that:

A. $\sup_{(i,j) \in E^2} P(i,j) < 1$

B. $\exists \sigma_0 < 1$ such that $\forall s > \sigma_0$,

$$\sup_{i \in E} \sum_{j \in E} P(i,j)^s < +\infty \quad \text{and} \quad \sum_{i \in E} \nu(i)^s < +\infty,$$

C. $\forall \epsilon > 0$ and $\forall s > \sigma_0$, $\exists A \subset E$ with $|A| < +\infty$ such that

$$\sup_{i \in E} \sum_{j \in E \setminus A} P(i,j)^s < \epsilon.$$

Then $\mathbf{X}$ satisfies the quasi-power property.

**Proof of the theorem**

**Lemma** If Assumptions A, B, C hold true, then $P_s : (\ell^1, \| \cdot \|_1) \to (\ell^1, \| \cdot \|_1)$ is a compact operator, $\forall s > \sigma_0$,

where $\ell^1 = \{ u = (u_i)_{i \in E} : \| u \|_1 = \sum_{i \in E} |u_i| < \infty \}$. 
We deduce from the lemma that the spectrum of $P_s$ is a sequence that converges to zero. Hence, $P_s$ has a finite number of eigenvalues with maximum modulus and there exists a spectral gap separating these dominant eigenvalues from the remainder of the spectrum.

Further, since $\mathbf{X}$ is ergodic, $P_s$ has a unique dominant eigenvalue $\lambda(s)$ which, moreover, is positive. Hence,

$$P_s^n u = \lambda(s)^n Q_s u + R_s^n u, \quad u \in \ell^1,$$

where $Q_s$ is the projector over the dominant eigenspace and $R_s$ is the projector over the remainder of the spectrum. The spectral radius of $R_s$ can be written $\rho(s) \cdot \lambda(s)$ with $\rho(s) < 1$.

Finally,

$$\Lambda_n(s) = \lambda(s)^n \|Q_s \nu_s\|_1 (1 + O(\rho(s)^n \lambda(s)^n)),$$

which means that $\mathbf{X}$ satisfies the quasi-power property.

The analyticity of the involved functions is due jointly to the analyticity of $s \to P_s$ and to perturbation arguments. \qed
**Theorem** Let $\mathbf{X}$ be a random sequence satisfying the quasi-power property with parameters $[\sigma_0, \lambda, c, \rho]$. Suppose that

$$
\phi(x) \sim c_1 \cdot x^s \cdot (\log x)^k \quad (P)
$$

with $s > \sigma_0$, $c_1 \in \mathbb{R}_+^*$ and $k \in \mathbb{N}^*$. Then the entropy rate $H_{h,\phi}(\mathbf{X})$ is given by the following table.

<table>
<thead>
<tr>
<th>Value of $s$</th>
<th>Condition on $h$</th>
<th>Entropy rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s = 1$</td>
<td>$h(x) \xrightarrow{x \to +\infty} c_2 \cdot x^{1/k}$</td>
<td>$c_2 \cdot c_1^{1/k} \cdot \lambda'(1)$</td>
</tr>
<tr>
<td></td>
<td>$h(x) = o(x^{1/k})$</td>
<td>$0$</td>
</tr>
<tr>
<td></td>
<td>$x^{1/k} = o(h(x))$</td>
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<tr>
<td>$s &gt; 1$</td>
<td>$h(x) \xrightarrow{x \to +\infty} c_2 \cdot \log x$</td>
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</table>
Proof \( \sup_{n_0 \in E^{n+1}} \nu_n(i_{n_0}^n) \to 0 \) and \( (P) \) together induce that \( \forall \epsilon > 0, \exists n_0 \in \mathbb{N} / n \geq n_0 \) and \( i_0^n \in E^{n+1} \),

\[
(1 - \epsilon)c_1 \nu_n(i_{0}^n)^s \log^k \nu_n(i_{0}^n) \leq \phi(\nu_n(i_{0}^n)) \\
\leq (1 + \epsilon)c_1 \nu_n(i_{0}^n)^s \log^k \nu_n(i_{0}^n),
\]

from which it follows that

\[
(1 - \epsilon)c_1 \Lambda_n^{(k)}(s) \leq \sum_{i_{0}^n \in E^{n+1}} \phi(\nu_n(i_{0}^n)) \leq (1 + \epsilon)c_1 \Lambda_n^{(k)}(s).
\]

Due to the analyticity of all involved functions,

\[
\Lambda_n^{(k)}(s) = c(s) \cdot \lambda'(s)^k \cdot n^k \cdot \lambda(s)^{n-k} \cdot [1 + O(1/n)].
\]

which yields

\[
\sum_{i_{0}^n \in E^{n+1}} \phi(\nu_n(i_{0}^n)) \sim c_1 \cdot c(s) \cdot \lambda'(s)^k \cdot n^k \cdot \lambda(s)^{n-k}.
\]

Since \( \phi \) is nonnegative, this sum converges polynomially to infinity. This leads to the next equivalences:

\[
\begin{align*}
h(\Sigma_n) & \sim c_2 \cdot |c_1|^{1/k} \cdot |\lambda'(1)| \cdot n \quad \text{if} \quad h(x) \sim c_2 \cdot x^{1/k}, \\
h(\Sigma_n) & \sim o(n) \quad \text{if} \quad h(x) = o(x^{1/k}), \\
h(\Sigma_n) & \sim s_n \cdot n \quad \text{with} \quad s_n \to \infty \quad \text{if} \quad x^{1/k} = o(h(x)).
\end{align*}
\]

Since by definition, the \((h, \phi)\)-entropy rate is the limit of \( h(\Sigma_n)/n \) when \( n \) tends to infinity, the results follow immediately for \( s = 1 \).

The other cases can be studied similarly. \( \square \)
<table>
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<tr>
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</tbody>
</table>

Values of classical entropy rates of a random sequence satisfying the quasi-power property with parameters $[\lambda, c, \rho, \sigma_0]$. 
Estimation of Shannon entropy rate for a finite Markov chain

For an ergodic **Markov chain** $X = (X_n)_{n \in \mathbb{N}}$ with state space $E$ with $s$ states, transition matrix $P = (P(i, j))$, where $P(i, j) = \mathbb{P}(X_{n+1} = j | X_n = i)$, and stationary distribution $\pi$ such that $\pi P = \pi$, and entropy

$$
\mathbb{H}(X) = - \sum_{i \in E} \pi(i) \sum_{j \in E} P(i, j) \log P(i, j) = h(P)
$$

$$
(= -\lambda'(1)).
$$

**Proposition** Anderson and Goodman (1957)
The empirical estimators

$$
\hat{P}_n(i, j) = \frac{\sum_{m=1}^{n} \mathbb{1}\{X_{m-1}=i, X_m=j\}}{\sum_{j \in E} \sum_{m=1}^{n} \mathbb{1}\{X_{m-1}=i, X_m=j\}}
$$

are strongly convergent and asymptotically normal:

$$
\sqrt{n} \left( \hat{P}_n(i, j) - P(i, j) \right) \xrightarrow{\mathcal{L}} \mathcal{N}_{s^2}(0, \Gamma^2)
$$

where $\Gamma^2_{ij} = \delta_{ik} \left[ \delta_{jl} P(i, j) - P(i, j) P(i, l) \right] / \pi(i)$.

- We define the plug-in estimator

$$
\hat{\mathbb{H}}_n = h(\hat{P}_n)
$$

of the entropy rate.
Theorem Ciuperca and Girardin (2007)

If the transition probabilities are not uniform, the plug-in estimator \( \hat{\mathbb{H}}_n = h(\hat{P}_n) \) of \( \mathbb{H}(X) \) is strongly convergent and asymptotically normal.

Precisely,

\[
\sqrt{n}[\hat{h}_n - \mathbb{H}(X)] \xrightarrow{L} \mathcal{N}(0, (\partial^i h) \Gamma(\partial^i h)'),
\]

where \( \partial^v u \) is the differential with order \( v \) with respect to variable \( u \) of \( h \).

Proof
Continuous mapping theorem and delta method
For a two-state chain
The transition matrix of the chain is
\[ P = \begin{pmatrix} 1 - p & p \\ q & 1 - q \end{pmatrix}. \]
The stationary distribution satisfies \( \pi P = \pi \), so
\[ \pi(0) = \frac{q}{p + q} \quad \text{and} \quad \pi(1) = \frac{p}{p + q}. \]
The entropy rate is
\[
\mathbb{H}(X) = h(p, q) = \pi(0)S_p + \pi(1)S_q \\
= \frac{q}{p + q}[-p \log p - (1 - p) \log(1 - p)] \\
+ \frac{p}{p + q}[-q \log q - (1 - q) \log(1 - q)].
\]
**Theorem** Girardin and Sesboue (2009)

\[ \hat{h}_n = h(p_n, \hat{q}_n) \xrightarrow{a.s.} \mathbb{H}(X). \]

If the chain is not uniform,

\[ \sqrt{n[\hat{h}_n - \mathbb{H}(X)]} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2) \]

where \( \sigma^2 = \Gamma(0, 0)^2[\partial_1^1 h(p, q)]^2 + \Gamma(1, 1)^2[\partial_2^1 h(p, q)]^2 \]

\[ = pq(1 - p) \left[ \frac{S_q - S_p}{p + q} - \log \frac{p}{1-p} \right] \]

\[ + pq(1 - q) \left[ \frac{S_p - S_q}{p + q} - \log \frac{q}{1-q} \right] \]

For illustration, we have simulated a chain for \( p = 0.2 \) and \( q = 0.3 \), for which \( \mathbb{H}(X) = 0.559 \).

The first figure shows the punctual convergence of \( \hat{h}_n \) to \( \mathbb{H}(X) \) for \( n = 10 \) to 5000 by steps of 10.

(Computation of \( \hat{h}_n \) for \( 10 \leq n \leq 5000 \) after simulation of one trajectory with length 5000)
This figure compares the empirical distribution function of $\sqrt{n}[\hat{h}_n - \mathbb{H}(X)]/\hat{\sigma}_n$ to that of the standard normal distribution for different values of $10 \leq n \leq 1000$. (for $T = 500$ trajectories simulated for each $n$)
Theorem Girardin and Sesboeue (2009)

For a uniform chain, \( p = q = 1/2 \), \( \hat{h}_n \) is strongly convergent and \( 2n[\mathbb{H}(X) - \hat{h}_n] \xrightarrow{\mathcal{L}} \chi^2(2) \).

Proof. \( \hat{h}_n - \mathbb{H}(X) = \)

\[
= [\partial_1^1 h(p, q)][\hat{P}(0, 1) - p] + [\partial_2^1 h(p, q)][\hat{P}(1, 0) - q] \\
+ \frac{1}{2} [\partial_1^2 h(p, q)][\hat{P}(0, 1) - p]^2 + \frac{1}{2} [\partial_2^2 h(p, q)][\hat{P}(1, 0) - q]^2 \\
+ o([\hat{P}(0, 1) - p]^2) + o([\hat{P}(1, 0) - q]^2) \\
= \frac{1}{2\Gamma(0, 0)^2} [\hat{\chi}^2(2)] + \frac{1}{2\Gamma(1, 1)^2} [\hat{\chi}^2(2)] \\
+ o([\hat{\chi}^2(2)] + o([\hat{\chi}^2(2)])
\]

and the result follows, since \( \frac{\sqrt{n}[\hat{\chi}^2(2)]}{\Gamma(0, 0)} \) and \( \frac{\sqrt{n}[\hat{\chi}^2(2)]}{\Gamma(1, 1)} \) are asymptotically standard normal. \( \square \)

The last figure compares the distribution function of \( 2n[\hat{h}_n - \mathbb{H}(X)] \) to that of the \( \chi^2(2) \)-distribution for \( n = 1000 \). (\( T = 1000 \) simulated trajectories for \( n \))
Estimation of generalized entropy rates

All the entropy rates are finite and non-zero only at a threshold where they are equal to the Rényi entropy rate up to a multiplicative factor. Therefore, we only estimate Shannon and Rényi entropy rates, that is

\[ h(\theta) = -\lambda'(1; \theta_0), \]

and \( h_s(\theta) = (1 - s)^{-1} \log \lambda(s; \theta_0). \)

The transition probabilities of the ergodic chain \( \mathbf{X} \) with denumerable state space are supposed to depend on \( \theta \in \Theta^r \), with true value \( \theta^0 \).

**Proposition** Billingsley (1962) Suppose that:

A. \( \forall x, \{y : P(x, y; \theta) > 0\} \) does not depend on \( \theta \).

B. \( \forall (x, y), P_u(x, y; \theta), P_{uw}(x, y; \theta) \) and \( P_{uvw}(x, y; \theta) \) are in \( C^1(\Theta) \).

C. \( \forall \theta \in \Theta, \exists N \), neighborhood such that \( \forall u, v, P_u(x, y; \theta) \) and \( P_{uw}(x, y; \theta) \) are uniformly bounded in \( L^1(\mu(dy)) \) on \( N \) and

\[ \mathbb{E}_\theta[\sup_{\theta' \in N} | P_u(x, y; \theta') |^2] < +\infty. \]

D. \( \exists \delta > 0 \) such that \( \mathbb{E}_\theta[| P_u(x, y; \theta) |^{2+\delta}] \) is finite \( \forall u = 1, \ldots, r \).

E. The Fisher information matrix

\[ \sigma(\theta) = (\mathbb{E}_\theta[P_u(x, y; \theta)P_v(x, y; \theta)]) \] is non-singular.

Then a strongly consistent maximum likelihood estimator \( \hat{\theta}_n \) of \( \theta \) exists. Moreover, \( \sqrt{n}(\hat{\theta}_u - \theta_u) \) is asymptotically normal, with covariance matrix \( \sigma^{-1}(\theta^0) \).
It is natural to consider the plug-in estimators:

\[ h(\hat{\theta}_n) = -\lambda'(1; \theta_n) \]

and \( h_s(\hat{\theta}_n) = (1 - s)^{-1} \log \lambda(s; \hat{\theta}_n) \)

of Shannon entropy rate and of Rényi entropy rate.

**Theorem** If Billingsley’s assumptions are satisfied and if \( \mathbf{X} \) satisfies the quasi-power property, then \( h(\hat{\theta}_n) \) and \( h_s(\hat{\theta}_n) \) are strongly consistent and asymptotically normal: \( \sqrt{n}[h(\hat{\theta}_n) - h(\theta)] \rightarrow \mathcal{N}(0, \Sigma_1) \), where

\[ \Sigma_1 = \left\{ \frac{\partial}{\partial \theta}[-\lambda'(1; \theta)] \right\}^t \sigma^{-1}(\theta) \frac{\partial}{\partial \theta}[-\lambda'(1; \theta)] \]

and \( \sqrt{n}[h_s(\hat{\theta}_n) - \mathbf{H}_s(\theta^0)] \rightarrow \mathcal{N}(0, \Sigma_s) \), where

\[ \Sigma_s = \frac{1}{(1 - s)^2} \left\{ \frac{\partial}{\partial \theta} \lambda(s; \theta) \right\}^t \sigma^{-1}(\theta) \frac{\partial}{\partial \theta} \lambda(s; \theta). \]

**Proof** Due to operators properties, the eigenvalue \( \lambda(s) \) and its derivative \( \lambda'(1) \) are continuous with respect to the perturbated operator \( P_s \). For a parametric chain depending on \( \theta \), Assumption B induces that \( P_s \) is a continuously differentiable function of \( \theta \). Therefore both \( \lambda(s; \theta) \) and \( \lambda'(s; \theta) \) are continuous with respect to \( \theta \). The results follow from the continuous mapping theorem and the delta method. \( \square \)
Estimation of the Entropy Rate of a Countable Markov Chain


Comparative Construction of Plug-in Estimators of the Entropy Rate of Two-State Markov Chains


Computation of Generalized Entropy Rates. Application and Estimation for Countable Markov Chains