Entropy Estimation for M/M/1 Queueing Systems

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Abstract. The aim of this paper is to estimate the entropy of Markovian queueing systems. The marginal entropy and entropy rate of a stochastic process are well-known to measure its uncertainty. When only observations of the process are available, the need to estimate them arises. Regnault [5, 6] provide estimators with good asymptotic properties of the entropy of continuous-time ergodic Markov processes. We specialize these results in the context of birth-death processes used in continuous-time M/M/1 queueing systems. Links with maximum entropy arguments are used to characterize the asymptotic behavior of the estimators.

Keywords: Queueing systems, birth-death processes, Shannon entropy, estimation

INTRODUCTION

The entropy of a random variable, as well as the entropy rate of a stochastic process, are well-known to measure their uncertainty. Both the entropy and the entropy rate have been originally introduced by Shannon [7] for independant and Markovian sequences of random variables in the context of information theory, and were proven to achieve the optimal channel capacity of Markovian communication systems. Since then, entropy rate has been introduced for other stochastic processes, such as finite state space continuous time Markov processes, for whom Bad Dumitrescu [2] obtained an explicit form.

Birth-death processes, are specially useful examples of Markov processes. Applied to queueing theory, they are of a particular interest in network models, reliability (see e.g., Englund [3]) or information sources (see e.g., Zamarlik [10]). M/M/1 queueing systems arise as particularly easy to handle examples of queueing systems. Due to the special form of their generator, their stationary distribution is easily computable which leads to simple expressions of both their marginal entropy (i.e., the entropy of their stationary distribution) and their entropy rate.

When only observations of a queueing system are available, the need to estimate entropy arises. Regnault [5] provides plug-in estimators of the marginal entropy of finite state space continuous time Markov processes assuming the observation of one long trajectory. The estimators are proven to be strongly consistent. The type of their asymptotic distribution is established, depending on whether the lagrangian of the marginal entropy, expressed as a function of the generator, is null or not. A characterization of the generators cancelling this lagrangian is obtained for two-state and three-state Markov processes. Furthermore, Regnault [6] provides plug-in estimators of the entropy rate of two-state Markov processes. They are proven to be strongly consistent with a normal asymptotic distribution except if the generator of the process is uniform.

The aim of this paper is to specialize and complete the results of Regnault [5, 6] in the context of birth-death processes associated to M/M/1 queueing systems. The stationary distribution, the marginal entropy and the entropy rate of such processes are expressed as explicit smooth functions of the birth and death rates. The strong consistency of the estimators is proven. The type of their asymptotic distribution is established and characterized in term of constraints on birth and death rates.

The paper is organized as follows. First, we recall necessary definitions about entropy and birth-death processes and establish explicit expressions of the stationary distribution, the marginal entropy and the entropy rate of birth-death processes associated to M/M/1 queueing systems. Second, we build plug-in estimators of both the marginal entropy and the entropy rate from the observation of one long trajectory of the process up to time. These estimators are proven to be strongly consistent and their asymptotic distributions are explicited.

M/M/1 QUEUEING SYSTEMS AND THEIR MARGINAL ENTROPY AND ENTROPY RATE

A queueing system is a system offering services to customers. The customers arrive one at a time and are served one at a time. Suppose the system has a reception capacity of s customers. Thus, any customer, finding s customers in the system when arriving, is lost. A queueing system is called a M/M/1 queueing system if, assuming that there are presently i customers in the system, for $0 \leq i \leq s$, the number of customers arriving next is modeled by a Poisson process with parameter λ_i, where $\lambda_i = \lambda > 0$ if $i < s$ and $\lambda_s = 0$, and the completion service time is exponentially distributed with parameter μ_i, where $\mu_i = \mu > 0$ if $i > 0$ and $\mu_0 = 0$. Then, the number of customers in the system is a birth-death Markov process $X = (X_t)_{t\in\mathbb{R}_+}$ with state space $\{0,\cdots,s\}$ and constant birth and death rates. It is a particular case of homogeneous finite state space Markov process. Hence, it satisfies the Markov property, i.e.,

$$\mathbb{P}(X_{s+t} = j | X_{s_1} = i_1,\cdots,X_{s_n} = i_n, X_s = i) = P_t(i,j),$$

where $n \in \mathbb{N}^*$, $s_1,\cdots,s_n,s,t \in \mathbb{R}^*$ such that $0 < s_1 < \cdots < s_n < s$ and $i_1,\cdots,i_n,i,j \in \{0,\cdots,s\}$. The function $t \in \mathbb{R}_+ \mapsto (P_t(i,j))_{(i,j)\in\{0,\cdots,s\}^2}$ is called the transition function of the process. It is differentiable at 0. Its derivative $A = (a_{i,j})_{(i,j)\in\{0,\cdots,s\}^2}$ is the generator of the process. It satifies the relations

$$\begin{array}{rcll}
a_{i,i-1} & = & \mu, & 0 < i \leq s, \\
a_{i,i+1} & = & \lambda & ,0 \leq i < s, \\
a_{i,i} & = & -\lambda-\mu, & i \in \{1,\cdots,s-1\}, \\
a_{o,o} & = & -\lambda, & \\
a_{s,s} & = & -\mu, & \\
a_{i,j} & = & 0, & |i-j| > 1.
\end{array}$$

Birth-death processes with constant birth and death rates are ergodic, i.e., the distribution of X_t tends to some unique distribution π called the stationary distribution of the process and satisfying $\pi A = 0$. Explicitely,

$$\pi_i = \left[\sum_{i=0}^{s}\left(\frac{\lambda}{\mu}\right)^i\right]^{-1}\left(\frac{\lambda}{\mu}\right)^i, \qquad i \in \{0,\cdots,s\}. \tag{1}$$

Note that π is the uniform distribution if $\lambda = \mu$.

Shannon [7] introduced the entropy of a probability distribution P with finite support E as $\mathbb{S}(P) = -\sum_{x\in E} P_x \log P_x$, with the convention $0\log 0 = 0$. The marginal entropy of a birth-death process is the Shannon entropy $\mathbb{S}(\pi) = -\sum_{i=0}^{s} \pi_i \log \pi_i$ of its stationary distribution. It measures the uncertainty of the process at equilibrium. Moreover, if X is stationary, i.e., if the initial distribution of the process is π, then the distribution of X_t at any time t is π and the marginal entropy is the entropy of X_t. From (1), we compute

$$\mathbb{S}(\pi) = S(\lambda,\mu) = \log\left(\sum_{i=0}^{s}\left(\frac{\lambda}{\mu}\right)^i\right) - \log\left(\frac{\lambda}{\mu}\right)\left[\sum_{i=0}^{s}\left(\frac{\lambda}{\mu}\right)^i\right]\left[\sum_{i=0}^{s} i\left(\frac{\lambda}{\mu}\right)^i\right]. \tag{2}$$

We will also consider the entropy of the process up to time T, defined as $H_T(X) = -\int f_T \log f_T dm$, where f_T is the likelihood of $(X_t)_{0\le t\le T}$ with respect to some reference measure m. The entropy up to time T of a birth-death process, divided by T, converges to a limit $\mathbb{H}(X)$ called the entropy rate of the process (see Bad Dumitrescu [2]), reprensenting its average uncertainty by unit of time. It is shown in Girardin and Sesboüé [4] that the entropy of birth-death processes takes the form

$$\mathbb{H}(X) = H(\lambda,\mu) = (2-\log(\lambda\mu))\left[\sum_{i=0}^{s-1}\lambda^i\mu^{s-i}\right]\left[\sum_{i=0}^{s}\lambda^i\mu^{s-i}\right]^{-1}. \tag{3}$$

ESTIMATION OF THE MARGINAL ENTROPY AND THE ENTROPY RATE

In this section, we assume that an M/M/1 queueing system has been observed up to time T. We compute plug-in estimators of both the marginal entropy and the entropy rate, by pluging estimators of λ and μ into (2) and (3).

Albert [1] computed the maximum likelihood estimator $\widehat{A}_T = (\widehat{A}_T(i,j))_{(i,j)}$ of the generator of a finite state space Markov process. Explicitely,

$$\widehat{A}_T(i,j) = \begin{cases} \frac{N_T(i,j)}{R_T(i)} & \text{if } R_T(i) \neq 0, \\ 0 & \text{else,} \end{cases} \quad , i \neq j$$

where $N_T(i,j)$ is the number of jumps from i to j observed along the trajectory and $R_T(i)$ the time spent at i by the trajectory. Albert [1] proved that $\widehat{A}_T$ converges almost surely to A as the time of observation T goes to infinity and that $\sqrt{T}\left(\widehat{A}_T - A\right)$ converges in distribution to a centered normal distribution. The asymptotic variance Σ_A^2 is diagonal with

$$\Sigma_A^2(a_{i,j}) = \frac{\rho a_{i,j}}{a^{(i,i)}}, \qquad (i,j) \in \{0,\cdots,s\}, i \neq j, \tag{4}$$

where ρ is the product of all non-zero eigenvalues of the generator and $a^{(i,i)}$ is its (i,i)-th cofactor.

In the special case of birth-death processes with constant birth and death rates, it yealds maximum likelihood estimators $\widehat{\lambda}_T$ and $\widehat{\mu}_T$ of λ and μ. Explicitely,

$$\widehat{\lambda}_T = \frac{1}{s}\sum_{i=0}^{s-1} \widehat{A}_T(i,i+1), \tag{5}$$

$$\widehat{\mu}_T = \frac{1}{s}\sum_{i=1}^{s} \widehat{A}_T(i,i-1). \tag{6}$$

Moreover, explicit expressions of the non-zero eigenvalues and cofactors of the generator are obtained below in Lemma 1, which in turn give in (4)

$$\begin{aligned} \Sigma_A^2(a_{i,i+1}) &= \frac{|\rho|\lambda}{\lambda^i \mu^{s-i}}, \qquad i \in \{0,\cdots,s-1\}, \\ \Sigma_A^2(a_{i,i-1}) &= \frac{|\rho|\mu}{\lambda^i \mu^{s-i}}, \qquad i \in \{1,\cdots,s\}, \end{aligned}$$

with

$$\rho = \begin{cases} \prod_{k=1}^{s/2} \left((\lambda+\mu)^2 - 2\lambda\mu\left(\cos\left(\frac{k\pi}{s+1}\right)\right)^2\right) & \text{if } s \text{ is even,} \\ -(\lambda+\mu)\prod_{k=1}^{(s-1)/2} \left((\lambda+\mu)^2 - 2\lambda\mu\left(\cos\left(\frac{k\pi}{s+1}\right)\right)^2\right) & \text{if } s \text{ is odd.} \end{cases} \tag{7}$$

Lemma 1 ***1.*** *The non-zero eigenvalues of the generator of a birth-death process with constant birth and death rates λ and μ are*

$$-\lambda - \mu + 2\sqrt{\lambda\mu}\cos\left(\frac{k\pi}{s+1}\right), \qquad k = 1,\cdots,s.$$

2. *The (i,i)-th cofactor of the generator is*

$$a^{(i,i)} = (-1)^s \lambda^i \mu^{s-i}, \qquad i \in \{0,\cdots,s\}.$$

Proof: 1. According to Yueh [9], the eigenvalues are of the form

$$-\lambda - \mu + 2\sqrt{\lambda\mu}\cos\theta, \tag{8}$$

with $\theta \in \mathbb{C}$ satisfying

$$\theta \neq m\pi, \qquad m \in \mathbb{Z}, \tag{9}$$

$$\text{and} \qquad \lambda\mu\left(\sin((s+2)\theta) + \sin(s\theta)\right) - (\lambda+\mu)\sqrt{\lambda\mu}\sin((s+1)\theta) = 0, \tag{10}$$

or equivalently, replacing $\sin((s+2)\theta)+\sin(s\theta)$ by $2\sin((s+1)\theta)\cos(\theta)$ in (10),

$$\sin((s+1)\theta)\left(2\lambda\mu\cos(\theta)-(\lambda+\mu)\sqrt{\lambda\mu}\right)=0.$$

Hence θ satisfies

$$\begin{aligned} \sin((s+1)\theta) &= 0, && (11)\\ \text{or} \qquad \cos(\theta) &= \frac{\lambda+\mu}{2}\frac{1}{\sqrt{\lambda\mu}}. && (12) \end{aligned}$$

Condition (12) induces the eigenvalue 0. Note that, in this case, θ may not be a real number since $(\lambda+\mu)/2 \geq \sqrt{\lambda\mu}$ and $\cos(\theta)\geq 1$.
Condition (11) is equivalent to $\theta = k\pi/(s+1)$, $k\in\mathbb{Z}$, which in turn in (8) together with (9) yields the non-zero eigenvalues $-\lambda-\mu+\sqrt{\lambda\mu}\cos(k\pi/(s+1))$, $k=1,\cdots,s$.
2. Straightforward computations. □

Proposition 1 *Let $X=(X_t)_{t\in\mathbb{R}_+}$ be the birth-death process associated to a M/M/1 queueing system with birth and death rates λ and μ. The couple $(\widehat{\lambda}_T,\widehat{\mu}_T)$ of estimators, defined in (5) and (6) from the observation of one trajectory up to time T, converges almost surely to (λ,μ) as T goes to infinity.*

Moreover, $\sqrt{T}(\widehat{\lambda}_T-\lambda,\widehat{\mu}_T-\mu)$ converges in distribution to a centered normal distribution with diagonal variance such that

$$\begin{aligned} \Sigma_\lambda^2 &= \frac{|\rho|\lambda}{s^2\mu^s}\sum_{i=0}^{s}\left(\frac{\mu}{\lambda}\right)^i, && (13)\\ \Sigma_\mu^2 &= \frac{|\rho|}{s^2\mu^{s-1}}\sum_{i=0}^{s}\left(\frac{\mu}{\lambda}\right)^i, && (14) \end{aligned}$$

with ρ given by (7).

Proof: According to Albert [1], $(\widehat{A}_T(i,i+1),\widehat{A}_T(i+1,i))_{i\in\{0,\cdots,s-1\}}$ converges almost surely to $(\lambda,\mu)^s$ and $\sqrt{T}(\widehat{A}_T(i,i+1)-\lambda,\widehat{A}_T(i+1,i)-\mu)_{i\in\{0,\cdots,s\}}$ converges in distribution to a centered normal distribution with variance Σ_A^2. Then, the almost sure convergence results from the continuous mapping theorem and the convergence in distribution results from the delta method (see e.g., Theorem 1.12 in Shao [8]), since $\widehat{\lambda}_T$ and $\widehat{\mu}_T$ are linear mappings of $\widehat{A}_T(i,i+1)$, $\widehat{A}_T(i+1,i)$, $i\in\{0,\cdots,s-1\}$. □

Estimation of the marginal entropy

Let us define the plug-in estimator of the marginal entropy of an M/M/1 queueing system by pluging the estimators $\widehat{\lambda}_T$ and $\widehat{\mu}_T$ into (2). Explicitely,

$$\widehat{S}_T = S(\widehat{\lambda}_T,\widehat{\mu}_T), \tag{15}$$

with s defined in (2)

This yealds strongly consistent estimators of the marginal entropy with explicit asymptotic distribution, as shown in Theorem 1.

Theorem 1 *Let $X = (X_t)_{t\in\mathbb{R}_+}$ be the birth-death process associated to an M/M/1 queueing system with birth and death rates λ and μ and asymptotic distribution π.*

1. *The plug-in estimator $\widehat{S}_T$ defined by (15) from the observation of one trajectory up to time T converges almost surely to the marginal entropy as T goes to infinity.*

2. *If $\lambda \neq \mu$, then $\sqrt{T}(\widehat{S}_T - \mathbb{S}(\pi))$ converges in distribution to a centered normal distribution with asymptotic variance*

$$\Sigma_s^2 = \left(\frac{\partial}{\partial\lambda}S\right)^2 \Sigma_\lambda^2 + \left(\frac{\partial}{\partial\mu}S\right)^2 \Sigma_\mu^2,$$

where Σ_λ^2, Σ_μ^2, $\frac{\partial}{\partial\lambda}S$ and $\frac{\partial}{\partial\mu}S$ are given by (16), (17), (13) and (14) respectively.

3. *If $\lambda = \mu$, $2sT(\mathbb{S}(\pi) - \widehat{S}_T)$ converges in distribution to a $\chi^2(1)$ distribution.*

Proof: Proposition 1 establishes the strong consistency of $(\widehat{\lambda}_T, \widehat{\mu}_T)$ and the asymptotic normality of $\sqrt{T}(\widehat{\lambda}_T - \lambda, \widehat{\mu}_T - \mu)$. Since the function S is twice continuously differentiable, the continuous mapping theorem applies and proves that $\widehat{S}_T$ converges almost surely to $\mathbb{S}(\pi)$. The delta method yields that $\sqrt{T}(\widehat{S}_T - \mathbb{S}(\pi))$ converges in distribution to a centered normal distribution if the derivative D_S of S is not null and $T(\mathbb{S}(\pi) - \widehat{S}_T)$ converges in distribution to a $\chi^2(1)$ distribution if D_S is null. Moreover, if $D_S \neq 0$, the asymptotic variance of $\sqrt{T}(\widehat{S}_T - \mathbb{S}(\pi))$ is $D_S.diag(\Sigma_\lambda^2, \Sigma_\mu^2).D_S^t$, where D_S^t denotes the transpose of D_S and $diag(\Sigma_\lambda^2, \Sigma_\mu^2)$ is the diagonal matrix with entries Σ_λ^2 and Σ_μ^2, defined by (13) and (14). It remains to show that $D_S = 0$ if and only if $\lambda = \mu$.
Differentiating (2) with respect to λ and μ yields

$$\frac{\partial}{\partial\lambda}S(\lambda,\mu) = \frac{1}{\mu}\log\left(\frac{\lambda}{\mu}\right)\left[\sum_{i=0}^{s}\left(\frac{\lambda}{\mu}\right)^i\right]^{-2}\left[\sum_{i,k=0}^{s}(k-i)i\left(\frac{\lambda}{\mu}\right)^{i+k-1}\right], \tag{16}$$

$$\frac{\partial}{\partial\mu}S(\lambda,\mu) = -\frac{\lambda}{\mu^2}\log\left(\frac{\lambda}{\mu}\right)\left[\sum_{i=0}^{s}\left(\frac{\lambda}{\mu}\right)^i\right]^{-2}\left[\sum_{i,k=0}^{s}(k-i)i\left(\frac{\lambda}{\mu}\right)^{i+k-1}\right]. \tag{17}$$

Let $\alpha = \lambda/\mu$ and $P(\alpha) = \sum_{i,k=0}^{s}(k-i)i\alpha^{i+k-1}$. Let us show that $P(\alpha)$ is never null. Setting $l = i+k$ and $m = k$, we obtain

$$P(\alpha) = \sum_{l=0}^{2s-1}\alpha^l.Q_l, \tag{18}$$

with

$$Q_l = \begin{cases} \displaystyle\sum_{m=0}^{l}(l-m)(2m-l) & \text{if } l \leq s, \\ \displaystyle\sum_{m=l-s}^{s}(l-m)(2m-l) & \text{if } l > s. \end{cases} \tag{19}$$

Let us show that $\sum_{m=0}^{l}(l-m)(2m-l)<0$ if $l\leq s$. A similar proof leads to $\sum_{m=l-s}^{s}(l-m)(2m-l)<0$ if $l>s$. Let us separate the terms in the sum according to whether $m\leq[l/2]$ or not, so that

$$\sum_{m=0}^{l}(l-m)(2m-l)=\begin{cases}\displaystyle\sum_{m=0}^{l/2-1}(l-m)(2m-l)+\sum_{m=l/2+1}^{l}(l-m)(2m-l) & \text{if } l \text{ is even,}\\ \displaystyle\sum_{m=0}^{(l-1)/2}(l-m)(2m-l)+\sum_{m=(l+1)/2}^{l}(l-m)(2m-l) & \text{if } l \text{ is odd.}\end{cases}$$

Changing m into $l-m$ in the sums $\sum_{m=l/2+1}^{l}(l-m)(2m-l)$ and $\sum_{m=(l+1)/2}^{l}(l-m)(2m-l)$ gives

$$\sum_{m=0}^{l}(l-m)(2m-l)=\begin{cases}\displaystyle -\sum_{m=0}^{l/2-1}(l-2m)^2<0 & \text{if } l \text{ is even,}\\ \displaystyle -\sum_{m=0}^{(l-1)/2}(l-2m)^2<0 & \text{if } l \text{ is odd.}\end{cases}\tag{20}$$

Hence, (20) together with (19) and (18) show that $P(\alpha)<0$, for all $\alpha>0$. Thus, according to (16) and (17), the derivative of s is null if and only if $\log(\lambda/\mu)=0$, i.e., if and only if $\lambda=\mu$. $\square$

Note that Regnault [6] proved that, for two-state and three-state Markov processes, the derivative D_s is null if and only if the stationary distribution is uniform (so that the entropy is at its maximum). Theorem 1 recovers this result in the special case of M/M/1 queueing systems with capacity 2 and 3 and extends it to larger capacities.

Estimation of the entropy rate

Let us define the plug-in estimator of the entropy rate of an M/M/1 queueing system by pluging the estimators $\widehat{\lambda}_T$ and $\widehat{\mu}_T$ into (3). Explicitely,

$$\widehat{H}_T=H(\widehat{\lambda}_T,\widehat{\mu}_T).\tag{21}$$

Theorem 2 *Let $X=(X_t)_{t\in\mathbb{R}_+}$ be the birth-death process associated to a M/M/1 queueing system with birth and death rates λ and μ and asymptotic distribution π.*

1. *The plug-in estimator $\widehat{H}_T$ defined by (21) from the observation of one trajectory up to time T converges almost surely to the entropy rate as T goes to infinity.*

2. *If the derivative D_h of h is not null, $\sqrt{T}(\widehat{H}_T-\mathbb{H}(X))$ converges in distribution as T goes to infinity to a centered normal distribution with asymptotic variance*

$$\Sigma_H^2=\left(\frac{\partial}{\partial\lambda}h\right)^2\Sigma_\lambda^2+\left(\frac{\partial}{\partial\mu}H\right)^2\Sigma_\mu^2,$$

where Σ_λ^2 and Σ_μ^2 are given by (13) and (14) respectively, and

$$\begin{aligned}
\frac{\partial}{\partial\lambda}H(\lambda,\mu) &= \left[\sum_{i=0}^{s}\left(\frac{\lambda}{\mu}\right)^i\right]^{-2}\left[\sum_{k=0}^{s}\sum_{i=0}^{s-1}\left(\frac{\lambda}{\mu}\right)^{i+k}((2-\log(\lambda\mu))(k+i+1)-i-1)\right],\\
\frac{\partial}{\partial\mu}H(\lambda,\mu) &= \left[\sum_{i=0}^{s}\left(\frac{\lambda}{\mu}\right)^i\right]^{-2}\left[\sum_{k=0}^{s}\sum_{i=0}^{s-1}\left(\frac{\lambda}{\mu}\right)^{i+k+1}((k-i-1)(2-\log(\lambda\mu))-1)\right].
\end{aligned}$$

3. *If $D_h = 0$, $2sT(\mathbb{H}(\pi)-\widehat{H}_T)$ converges in distribution as T goes to infinity to a $\chi^2(2)$ distribution.*

Proof: In the same way as in the proof of Theorem 1, the continous mapping theorem and the delta method applied to $(\widehat{\lambda}_T,\widehat{\mu}_T)$ yield the strong consistency of $\widehat{H}_T$, the convergence in distribution of $\sqrt{T}(\widehat{H}_T-\mathbb{H}(X))$ to a centered normal distribution with variance $\Sigma_H^2 = D_H diag(\Sigma_\lambda^2,\Sigma_\mu^2)D_H^t$ if $D_H \neq 0$ and the convegence of $2sT(\mathbb{H}(X)-\widehat{H}_T)$ to a $\chi^2(2)$ distribution if $D_H = 0$.

$\square$

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